Intergenerational Insurance

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Abstract

How should successive generations insure each other when the enforcement of transfers between them is limited? This paper examines transfers that maximize the expected discounted utility of all generations subject to a participation constraint for each generation. The resulting optimal intergenerational insurance is history dependent even when the environment is stationary. Consequently, consumption is heteroskedastic and autocorrelated across generations. The optimal intergenerational insurance arrangement is interpreted as a pay-as-you-go social security scheme with means testing and a mixture of flat-rate and contributory-related elements. With logarithmic preferences, the pension received when old depends on the contribution rate paid when young.

Keywords: Intergenerational insurance; limited commitment; risk sharing; social security; stochastic overlapping generations.

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Introduction

Countries face economic shocks that result in unequal exposure to risk across generations. The Financial Crisis of 2008 and the Covid-19 pandemic are two recent and notable examples.[1] Confronted with such shocks, it is desirable to share risk across generations through a social security scheme or another form of insurance. However, full risk sharing is not sustainable if it commits future generations to transfers they would not wish to make once they are born. The issue of the sustainability of social security insurance is becoming increasingly relevant in many advanced economies as the relative standard of living of the younger generation has worsened in recent decades.[2] If this generational shift persists, then future generations may be less willing to contribute to social security arrangements than in the past. Therefore, a natural question to ask is, how should an optimal intergenerational insurance arrangement be structured when there is limited enforcement of risk-sharing transfers?

Despite its policy relevance, this question has not been fully addressed by the literature on intergenerational insurance. The normative approach in the literature investigates the optimal design of intergenerational insurance but neglects the limited enforcement of risk-sharing transfers by assuming that transfers are mandatory. Meanwhile, the positive approach highlights the political limits to intergenerational insurance, while considering equilibrium allocations supported by a particular voting mechanism that are not necessarily Pareto optimal.

In this paper, we examine optimal intergenerational insurance when the enforcement of risk-sharing transfers is limited. Limited enforcement is modeled by assuming that transfers satisfy a participation constraint for each generation. This can be interpreted as requiring that the insurance arrangement be supported by each generation if put to a vote. An arrangement of risk-sharing transfers is *sustainable* if it satisfies the participation constraint of every generation. *Optimal* sustainable intergenerational insurance is determined by a benevolent social planner who chooses transfers to maximize the expected discounted utility of all generations subject to the participation constraints.

[1] Glover et al. (2020) find that the Financial Crisis of 2008 had a greater negative impact on the older generation, while the young benefited from the fall in asset prices. Glover et al. (2021) find that younger workers have been impacted to a greater extent by the response to the Covid-19 pandemic because they disproportionately work in sectors that have been particularly adversely affected, such as retail and hospitality.

[2] Part A of the Supplementary Appendix reports the changes in the relative standard of living of the young and the old for six OECD countries using data from the Luxembourg Income Study Database.

The model is simple. There is a representative agent in each generation and a single, non-storable consumption good. Agents live for two periods: young and old. The endowments of both the young and the old are stochastic. The underlying economy is stationary and the shock to endowments is identically and independently distributed over time, though there may be aggregate as well as idiosyncratic risk. There is no population growth, no production, no altruism, and no asymmetry of information. There are only two frictions. First, risk may not be allocated efficiently, even if the economy is dynamically efficient, because there is no market in which the young can share risk with previous generations (see, for example, Diamond, 1977). Second, the amount of risk that can be shared is limited because transfers between generations cannot be enforced. In particular, the old will not make a transfer to the young (since the old have no future) and the young will only make a transfer to the old if the promise made to them for their old age at least matches their expected lifetime utility from autarky and they anticipate that the promise will be honored by the next generation.

It is well known (see, for example, Aiyagari and Peled, 1991) that if endowments are such that the young wish to defer consumption to old age at a zero net interest rate, then there are stationary transfers that improve upon autarky (Proposition 1). Under this condition, and provided that the first-best transfers cannot be sustained, we find that there is a trade-off between efficiency and providing incentives for the young to make transfers to the old. This trade-off is resolved by linking the utility that the young are promised for their old age to the promise made to the young of the previous generation. The resulting optimal sustainable intergenerational insurance arrangement is history dependent, even though the environment itself is stationary. Consequently, the risk from an endowment shock is unevenly spread into the future, generating heteroskedasticity and autocorrelation of consumption.

To understand why there is history dependence, suppose that the first-best transfers would violate the participation constraint of the young in some endowment state. To ensure that the current transfer made by the young is voluntary, either the current transfer is reduced below the first-best level, or the promised transfers for their old age are increased. Both changes are costly since a smaller current transfer reduces the amount of risk shared today, while increasing the transfers promised to the current young for their old age tightens the participation constraints of the next generation and reduces the risk that can be shared tomorrow. Therefore, there is an optimal trade-off between reducing the current transfer and increasing the future promised utility, which depends both on the current endowment and the current promise. For example, consider some current endowment and current promise such that the future promise is above the current one. If the same endowment state is repeated in the subsequent period, then the young in that period are called upon to make a larger transfer, which in turn requires a higher promise of future utility to them as well. Thus, the transfer depends not only on the current endowment but also on the history of endowment shocks.

The optimal sustainable intergenerational insurance is found by solving a functional equation derived from the planner's maximization problem. The solution is characterized by policy functions for the current transfer made by the young (or equivalently, the consumption of the young) and the future promised utility for their old age. Both policy functions depend on the endowment state and the current promise and are weakly increasing in the current promise for a given endowment state (Lemmas 2 and 3). There is a unique fixed point for the future promise that depends on the first-best transfer in that endowment state. For a given endowment state, the future promised utility is increased when the current promise is less than the corresponding fixed point and it is decreased when the current promise is greater than the fixed point. When the promised utility is sufficiently low, there is some endowment state in which the participation constraint of the young does not bind. In that case, the future promise is *reset* to the largest value that maximizes the planner's payoff.

The resetting property shows that the effect of a shock does not last forever. Moreover, it implies strong convergence to a unique and non-degenerate invariant distribution of promised utilities (Proposition 4). Since the invariant distribution of promised utility is non-degenerate and the optimal insurance arrangement is history dependent, consumption fluctuates across states and over time, even in the long run. This is in stark contrast to the situation with either full enforcement of transfers or no risk. In the former case, the promised utility is constant over time, except possibly in the initial period (Proposition 2). In the latter case, the promised utility is constant in the long run, although there may be a finite initial phase during which the promised utility falls (Proposition 3). In either case, there is no inefficiency in the long run. Thus, *both* risk and limited enforcement are necessary for the long-run distribution of promised utility to be non-degenerate and for there to be inefficiency in the long run.

We use measures of entropy (see, for example, Backus, Chernov, and Zin, 2014) and the bound on the variability of the implied yields introduced by Martin and Ross (2019) to understand how risk is shared across generations. These risk measures are derived from the state prices and implied yields that correspond to the optimal sustainable intergenerational insurance. The implied yields increase with the current promise, indicating that generations born in a period with a higher promise bear greater risk. Moreover, the yield on all very long bonds converges to a long-run yield determined by the Perron root of the state price matrix, indicating that the exposure to a shock dies out as the time horizon becomes long enough. For some parameter values, the long-run yield is equal to the planner's discount factor (Proposition 5). We present an example with two endowment states, the solution to which can be computed using a simple shooting algorithm without the need to solve a functional equation. We provide a closed-form solution for the bound on the variability of the implied yields (Proposition 6) and show that the invariant distribution of promised utility is a transformation of a geometric distribution (Proposition 7).

In addition, we provide an interpretation of the optimal sustainable intergenerational insurance in terms of a pay-as-you-go social security or state pension system. We show how the pension received in old age depends on a measure of the contribution made when young. In the case of logarithmic preferences, this measure is simply the contribution rate. In this case, the pension is independent of the contribution rate paid when young if that contribution rate is below a critical threshold. Otherwise, it is an increasing function of the past contribution rate. For a given contribution rate, the pension also depends on current endowments. If, for example, the aggregate endowment is fixed, then for each given contribution rate paid when young, the pension the old receive is inversely related to their own endowment. That is, the optimal pension scheme will have both means testing and a mixture of flat-rate and contributory-related elements. All three of these features are commonly found in national pension schemes.[3] Absent enforcement issues, the optimal pension scheme has only flat-rate and means-testing elements and therefore, a key prediction is that the optimal pension scheme has a contributory-related element whenever there is limited enforcement of transfers.

Literature. — The existing literature on risk sharing in overlapping generations models has several strands. One strand considers public policies or other non-market mechanisms that improve risk sharing through a social security scheme (see, for example, Enders and Lapan, 1982; Shiller, 1999; Rangel and Zeckhauser, 2000). In this strand of the literature, however, transfers are mandatory and consideration is restricted to stationary transfers, in contrast to the voluntary and history-dependent transfers considered here. Our result

[3] Conde-Ruiz and Profeta (2007) discuss how to classify pay-as-you-go state pension systems. They distinguish between Bismarckian systems, where there is a strong link between pensions and contributions, and Beveridgean systems, where the link is weaker. Using observed correlations between pension income and pre-retirement earnings, they find that Austria, France, Germany, Greece, Italy and Spain are (to varying degrees) Bismarckian, whereas Belgium, Denmark, Ireland, the United Kingdom and the United States are more Beveridgean. They also report that Austria, Belgium, France, Ireland, Italy, the United Kingdom and the United States have means testing, whereas Denmark, Germany, Greece and Spain do not.

on history dependence is foreshadowed in a mean-variance setting by Gordon and Varian (1988), who establish that any time-consistent optimal intergenerational risk-sharing agreement is non-stationary. Ball and Mankiw (2007) consider how risk is allocated across generations in a complete-markets equilibrium in which all generations can trade contingent claims before they are born. They find that shocks are evenly spread across generations in an optimal allocation and hence, consumption follows a random walk. This allocation is not sustainable because it implies that the participation constraint of some future generation is violated almost surely. In contrast, we show that shocks are unevenly spread across future generations to ensure that all participation constraints are met.[4]

A second strand of the literature provides simple necessary and sufficient criteria for Pareto optimality. Aiyagari and Peled (1991) derive a dominant root condition for interim optimality in an endowment economy with a finite state space. This approach has been extended by several authors (see, for example, Manuelli, 1990; Chattopadhyay and Gottardi, 1999; Demange and Laroque, 1999; Bloise and Calciano, 2008). Labadie (2004) shows how to interpret this characterization in terms of ex ante Pareto optimality. We provide a similar characterization in which the Pareto weights are determined endogenously by the participation constraints and the history of shocks.

A third strand of the literature examines political economy models of social security in which agents vote on the tax and benefit rates for intergenerational transfers (see, for example, Cooley and Soares, 1999; Boldrin and Rustichini, 2000; Bassetto, 2008; Gonzalez-Eiras and Niepelt, 2008). A social security scheme corresponds to the subgame (or Markov) perfect equilibria of a repeated (or dynamic) game. Typically, it is assumed that voters in the working-age group are pivotal. In equilibrium, a social security scheme is supported by current workers in the expectation that future workers will do the same. The equilibria of these games are not necessarily Pareto optimal. In contrast, the approach presented here identifies constrained Pareto optimal intergenerational transfers that each generation unanimously agrees to respect.

The paper is methodologically related to the literature on risk sharing and limited commitment with infinitely-lived agents. This literature examines two polar cases: one with two infinitely-lived agents (see, for example, Thomas and Worrall, 1988; Chari and Kehoe, 1990; Kocherlakota, 1996) and the other with a continuum of infinitely-lived

[4] Although not directly focused on risk sharing, two recent papers do study overlapping generations models with limited commitment. Dovis, Golosov, and Shourideh (2016) study fiscal policy in an open economy where an infinitely-lived government can default on external debt or redistribute wealth through transfers once investment decisions have been made. Kiyotaki and Zhang (2018) examine a firm's investment in worker training when the worker cannot commit to stay with the firm in the next period.

agents (see, for example, Thomas and Worrall, 2007; Krueger and Perri, 2011; Broer, 2013). The overlapping generations model considered here has a continuum of agents but only two agents are alive at any point in time. The model is not nested in either of the two infinitely-lived agent models but fills an important gap by providing an analysis of optimal intergenerational insurance with limited commitment.

The paper has the following structure. Section I sets out the model. Section II considers two benchmarks: one with full enforcement of transfers from the young to the old and the other without risk. Section III characterizes the optimal sustainable intergenerational insurance and provides an interpretation of the solution as a pay-as-you-go social security or state pension system. Section IV establishes convergence to an invariant distribution on a countable ergodic set. Section V considers how risk is allocated at the invariant distribution. Section VI studies a case with two endowment states. Section VII considers social security schemes that are easily implementable and compares the welfare and risk properties relative to the optimum. Section VIII discusses the results and some extensions of the basic model. Section IX concludes. The Appendix contains the proofs of the main results. Additional proofs and further details can be found in the Supplementary Appendix.

I The Model

Time is discrete and indexed by $t = 0, 1, 2, \ldots, \infty$. The model consists of a pure exchange economy with an overlapping generations demographic structure. At each time t, a new generation is born and lives for two periods. Each generation is composed of a single agent.[5] The agent is young in the first period of life and old in the second. The economy starts at $t = 0$ with an initial old agent and an initial young agent. Since time is infinite, the initial old agent is the only agent that lives for just one period.

At each t, agents receive an endowment of a perishable consumption good. Endowments are finite and strictly positive and depend on the state of the world $s_t \in \mathcal{S} := \{1, 2, \ldots, S\}$ with $S \geq 2$. The endowments of the young and the old in state s_t are $e^y(s_t)$ and $e^o(s_t)$, and the aggregate endowment is $e(s_t) := e^y(s_t) + e^o(s_t)$. Denote the history of states up to and including time t by $s^t := (s_0, s_1, ..., s_t) \in \mathcal{S}^t$ and the probability of reaching history s^t by $\pi(s^t) = \pi(s^{t-1})\pi(s_t \mid s^{t-1})$. We assume that states are identically and independently distributed (hereafter, i.i.d.). Hence, $\pi(s^t) = \pi(s_0) \cdot \ldots \cdot \pi(s_t)$ where $\pi(s_t)$ is the probability of state s_t and $\pi(s_t \mid s^{t-1}) = \pi(s_t)$. There is complete information: all

[5] The assumption that there is a representative agent in each generation makes it possible to focus on intergenerational risk sharing. By doing so, however, we ignore questions about inequality within generations and its evolution over time.

information about endowments and the probability distribution is public. Let $c^y(s^t)$ and $c^o(s^t)$ be the per-period consumption of the young and the old. There is no technology for saving or investment and hence, $c^y(s^t) + c^o(s^t) = e(s_t)$. Endowments depend only on the current state whereas consumption can, in principle, depend on the history of states. In autarky, agents consume only their own endowments, that is, $c^y(s^{t-1}, s_t) = e^y(s_t)$ and $c^o(s^{t-1}, s_t) = e^o(s_t)$ for all t and (s^{t-1}, s_t).

Each generation is born after that period's uncertainty is resolved when the current endowments of the young and the old are known. Therefore, after birth, a generation only faces uncertainty in old age and there is no insurance market in which the young can insure against their endowment risk. The lifetime endowment utility of an agent born in state s_t is:

$$\hat{v}(s_t) := u(e^y(s_t)) + \beta \sum\nolimits_{s_{t+1}} \pi(s_{t+1}) u(e^o(s_{t+1})),$$

where $\beta \in (0, 1]$ is the generational discount factor and $u(\cdot)$ is the utility function, common to both the young and the old. Since endowments are positive and finite, $\hat{v}(s_t)$ is bounded.

Assumption 1. The utility function $u\colon \mathbb{R}_+ \to \mathbb{R} \cup \{-\infty\}$ is strictly increasing, strictly concave, thrice continuously differentiable, satisfies the Inada conditions and

$$u(0) < \min_s \left\{ u(e^y(s)) - \beta \sum\nolimits_{s'} \pi(s') \left(u(e(s')) - u(e^o(s')) \right) \right\},$$

where s' is the endowment state at the next date.

The latter part of Assumption 1 is sufficient to guarantee that young agents have positive consumption at the optimal sustainable intergenerational insurance. Assumption 1 is satisfied when $\lim_{c \to 0} u(c) = -\infty$, as is the case when the utility function is logarithmic or has constant coefficient of relative risk aversion greater than one.

Let $\lambda(s) := u_c(e^y(s))/u_c(e^o(s))$ denote the ratio of marginal utilities of the young and the old in autarky in state s. If $\lambda(s) > 1$, then it follows from the assumption that the utility of consumption is independent of age that $e^y(s) < e^o(s)$ and the old are wealthier than the young in that state. Order states such that $\lambda(S) \geq \lambda(S-1) \geq \cdots \geq \lambda(1)$. Then, the relative endowment of the old, compared to the young, increases with the state.[6] Since $\lambda(s)$ varies across states, it is desirable to share risk across generations. In the absence

[6] When two states have the same value of $\lambda(\cdot)$, we use the convention that the states are ordered by the aggregate endowment, that is, higher states are associated with a higher aggregate endowment. A special case is where states can be ordered so that the endowment of the old is increasing in s, while the endowment of the young is decreasing in s.

of a storage technology and because the young are born after uncertainty is resolved, the only possibility for intergenerational insurance is through transfers between the young and the old. However, we require that all transfers must be voluntary. That is, agents make a transfer only if it is in their interest to do so. We assume that any generation that does not make a transfer when called upon to do so receive no transfer when they reach old age. Therefore, for every history of shocks, intergenerational insurance must provide all generations with at least the same lifetime utility they derive from consuming their endowments.

Benevolent Social Planner. — Consider the problem of a benevolent social planner who chooses an intergenerational insurance rule, that is, a function $\tau(s^t)$ that specifies the transfer between the young and the old for each history s^t. Since the aggregate endowment is consumed, $\tau(s^t) = e^y(s_t) - c^y(s^t)$ and $c^o(s^t) = e(s_t) - c^y(s^t)$. The planner must respect the constraint that neither the old nor the young would be better off in autarky than adhering to the specified transfers. Therefore, the transfer from the young to the old is always non-negative because the old would default if they were ever called upon to make a transfer. The non-negativity constraint is subsumed by requiring $c^y(s^t) \in \mathcal{Y}(s_t) := [0, e^y(s_t)]$ for every history s^t. The analogous participation constraint for the young requires that transfers received in old age sufficiently compensate a transfer made when young so that the agent is no worse off than reneging on the transfer today and receiving the corresponding autarkic utility. That is,

$$u(c^y(s^t)) + \beta \sum\nolimits_{s_{t+1}} \pi(s_{t+1}) u(e(s_{t+1}) - c^y(s^t, s_{t+1})) \geq \hat{v}(s_t) \qquad \forall\, s^t. \tag{1}$$

Then, $\Lambda := \{\{c^y(s^t) \in \mathcal{Y}(s_t)\}_{t=0}^{\infty} \mid (1)\}$ is the planner's constraint set. Since utility is strictly concave and the constraints in (1) are linear in utility, Λ is convex and compact.

Definition 1. An Intergenerational Insurance rule is sustainable if the history-dependent sequence $\{c^y(s^t)\}_{t=0}^{\infty} \in \Lambda$.

The planner seeks to address the conflict between generations by choosing a sustainable intergenerational insurance rule that maximizes a weighted sum of the expected utilities of all generations. We suppose that the planner discounts the expected utility of future generations by a factor $\delta \in (0, 1)$ and weighs the utility of the initial old by β/δ.[7] We allow the planner's discount factor, δ, to differ from the generational discount factor, β.

[7] The assumption of geometric discounting for the planner is common (see, for example, Farhi and Werning, 2007). Using a weight of β/δ for the initial old preserves the same relative weights on the young and the old in every period.

Definition 2. A Sustainable Intergenerational Insurance rule is optimal if it maximizes the weighted sum of the expected utilities of all generations given by:

$$\frac{\beta}{\delta}\sum_{s_0}\pi(s_0)u(e(s_0)-c^y(s_0))+\mathbb{E}_0\left[\sum_t\delta^t\left(u(c^y(s^t))+\beta\sum_{s_{t+1}}\pi(s_{t+1})u(e(s_{t+1})-c^y(s^t,s_{t+1}))\right)\right], \quad (2)$$

where $\mathbb{E}_0$ is the expectation over all histories of endowment states, subject to the constraint that the initial old receive an expected utility of at least ω:

$$\sum\nolimits_{s_0}\pi(s_0)u(e(s_0)-c^y(s_0)) \geq \omega. \quad (3)$$

Let $V(\omega)$ denote the value function corresponding to a solution of the optimization problem in Definition 2. The function $V(\omega)$ traces out the Pareto frontier between the expected utility of the initial old and the expected discounted utility of all future generations.[8] It is defined on a set $\Omega := [\omega_{\min}, \omega_{\max}]$ where $\omega_{\min} := \sum_{s_t}\pi(s_t)u(e^o(s_t))$ is the expected autarkic utility of the old and $\omega_{\max}$ is the highest feasible value of expected utility of the old consistent with the participation constraints ($\omega_{\max}$ is discussed in more detail below). Let

$$\omega_0 := \sum\nolimits_{s_0}\pi(s_0)u(e(s_0)-\tilde{c}^y(s_0)), \quad (4)$$

where the notation $\tilde{c}^y(s_0)$ is adopted to emphasize that it is part of the optimal solution. Clearly, for $\omega \leq \omega_0$, constraint (3) does not bind and $V(\omega) = V(\omega_0)$; whereas for $\omega > \omega_0$, constraint (3) binds and $V(\omega) < V(\omega_0)$.

Preliminaries. — The existence of a non-autarkic sustainable allocation can be addressed by considering small stationary transfers (that is, transfers depending only on the current endowment state). Denote the intertemporal marginal rate of substitution between consumption when young in state s and consumption when old in state s', evaluated at autarky, by $\hat{m}(s,s') := \beta u_c(e^o(s'))/u_c(e^y(s))$ and let $\hat{q}(s,s') := \pi(s')\hat{m}(s,s')$. The terms $\hat{m}(s,s')$ and $\hat{q}(s,s')$ correspond to the stochastic discount factor and the state prices in an equilibrium model. Denote the matrix of terms $\hat{q}(s,s')$ by $\hat{Q}$. A non-autarkic sustainable allocation exhausting the aggregate endowment and satisfying $c^y(s^t) \in \mathcal{Y}(s_t)$ and the participation constraints in (1) exists whenever the Perron root (the leading eigenvalue) of $\hat{Q}$ is greater than one (see, for example, Aiyagari and Peled, 1991; Chattopadhyay and Gottardi, 1999). In this case, there is a vector of strictly positive sta-

[8] More precisely, the function $\tilde{V}(\omega) := V(\omega) - \omega$ can be viewed as a Pareto frontier that trades off the expected utility of the initial old against the planner's valuation of the expected discounted utility of all other generations.

tionary transfers that improve the lifetime utility of the young in each state. Since the endowment states are transitory, the matrix $\hat{Q}$ has rank one and the Perron root is equal to the trace of the matrix. We assume that this trace is larger than one.

ASSUMPTION 2. $\beta \sum_s \pi(s)\lambda(s)^{-1} > 1$.

If there is just one state, then Assumption 2 reduces to the standard Samuelson condition: $\beta u_c(e^o) > u_c(e^y)$. In this case, it is well known that there are Pareto-improving transfers from the young to the old. Assumption 2 is the generalisation to the stochastic case and a natural assumption given that our focus is on transfers to the old.[9] Given Assumption 2, it follows that the constraint set Λ is non-empty.

PROPOSITION 1. Under Assumption 2, there exists a non-autarkic and stationary Sustainable Intergenerational Insurance rule.

Since $\lambda(s)$ is increasing in s, Assumption 2 implies that $\beta > \lambda(1)$, or equivalently, $\beta u_c(e^o(1)) > u_c(e^y(1))$. That is, the state-wise Samuelson condition is satisfied in state 1. We shall assume that the opposite is true in state S.

ASSUMPTION 3. $\lambda(S) \geq \beta/\delta$.

Since $\delta < 1$, Assumption 3 implies that $\beta < \lambda(S)$, or equivalently, $\beta u_c(e^o(S)) < u_c(e^y(S))$. We make Assumption 3 for two reasons. First, it shows that the analysis below does not depend on the state-wise Samuelson condition applying in every state. Second, it provides a simple sufficient condition for the strong convergence result of Section IV.

II Two Benchmarks

Before turning to the characterization of the optimal sustainable intergenerational insurance, it is helpful to consider two benchmark cases that serve to illustrate the inefficiencies generated by the presence of limited enforcement and uncertainty. The first benchmark ignores the participation constraints of the young but not the participation constraints of the old. The second benchmark has only one endowment state but requires that the planner respects the participation constraints of both the young and the old.

[9] A simple sufficient condition for Assumption 2 to be satisfied is that the Frobenius lower bound, given by the minimum row sum of $\hat{Q}$, is greater than one. That is, $\sum_{s'} \hat{q}(s,s') > 1$ for each state s. This implies that in autarky the young would, if they could, save for their old age in each endowment state, even at a zero net rate of interest.

First Best. — We assume there is uncertainty, $S \geq 2$, but suppose that the planner ignores the participation constraints of the young. Let $\Lambda^* := \{c^y(s^t) \in \mathcal{Y}(s_t)\}_{t=0}^{\infty}$ denote the set of transfers without the constraints in (1).[10]

Definition 3. An Intergenerational Insurance $\{c^y(s^t)\}_{t=0}^{\infty} \in \Lambda^*$ is first best if it maximizes the objective function (2) subject to constraint (3).

It is easy to verify that at the first-best optimum:

$$\frac{u_c(c^{y*}(s^t))}{u_c(e(s_t) - c^{y*}(s^t))} = \max\left\{\frac{\beta}{\delta}, \lambda(s_t)\right\} \qquad \forall\, s^t. \tag{5}$$

Condition (5) is the familiar Arrow-Borch condition for optimal risk sharing modified to account for the constraint that transfers are only from the young to the old. It shows that $c^{y*}(s^t)$ is independent of the history s^{t-1} and depends only on the current state s_t, that is, transfers are stationary. Let $\tau^*(s) = e^y(s) - c^{y*}(s)$ denote the first-best transfer conditional on state s. The transfer $\tau^*(s) = 0$ for states in which the participation constraint of the old binds, that is, for states in which $\beta/\delta \leq \lambda(s)$. Under Assumption 3, there is always one such state and hence, the first-best transfer is not positive in every state. The transfer $\tau^*(s)$ is positive for states in which $\beta/\delta > \lambda(s)$. For such states, condition (5) shows that the ratio of marginal utilities is constant across each of these states and across all generations. It can be seen from condition (5) that for states in which transfers are positive, $\tau^*(s)$ is increasing in β since a higher β puts more weight on the utility of the old who receive the transfer, whereas $\tau^*(s)$ is decreasing in δ since a higher δ puts more weight on the utility of the young who make the transfer.

Let $\omega^* := \sum_s \pi(s)u(e(s) - c^{y*}(s))$ denote the expected utility of the old in the first-best solution. From the definition in (4), it follows that $\omega_0 = \omega^*$. Now consider constraint (3). For $\omega \leq \omega^*$, constraint (3) does not bind and the first-best consumption at time $t = 0$, $c^{y*}(s_0)$, is determined by condition (5), as in every other time $t > 0$. For $\omega > \omega^*$, constraint (3) binds and the initial transfers to the old are correspondingly higher. In particular, if $\omega > \omega^*$, then there is a $\nu_0 > 0$ (the multiplier associated with constraint (3)

[10] Hereafter, the asterisk designates the first-best outcome. Note that the first best could be defined by assuming that the planner ignores the participation constraints of both the young and the old. The reason for presenting the first best as we do is to show that this allocation is stationary. Hence, any history dependence in the optimal sustainable intergenerational insurance rule derives from the imposition of the participation constraints of the young.

is $(\beta/\delta)\nu_0)$ such that consumption at $t = 0$ satisfies:

$$\frac{u_c(c^{y*}(s_0))}{u_c(e(s_0) - c^{y*}(s_0))} = \max\left\{\frac{\beta}{\delta}(1+\nu_0), \lambda(s_0)\right\} \qquad \forall\, s_0, \tag{6}$$

and constraint (3) holds with equality. Again, for states in which the transfers are positive, the ratio of the marginal utilities is constant.

Denote the per-period payoff to the planner with the first-best allocation by $v^* := \sum_s \pi(s)u(c^{y*}(s)) + (\beta/\delta)\omega^*$ and the expected discounted payoff to the planner for $\omega \in \Omega$ by $V^*(\omega)$. The first-best outcome is summarized in the following proposition.[11]

PROPOSITION 2. (i) The consumption of the young $c^{y*}(s^t)$ is stationary and satisfies condition (5) for $t > 0$ and condition (6) for $t = 0$. (ii) The value function $V^*\colon \Omega \to \mathbb{R}$ is equal to $V^*(\omega) = v^*/(1-\delta)$ for $\omega \in [\omega_{\min}, \omega^*]$ and is strictly decreasing and strictly concave for $\omega \in (\omega^*, \omega_{\max}]$ with $\omega_{\max} := \sum_s \pi(s)u(e(s))$ and $\lim_{\omega\to\omega_{\max}} V^*_\omega(\omega) = -\infty$.

Note that when $\omega > \omega^*$, the consumption of the young is lower than the first-best consumption given by condition (5), but only in the initial period. In terms of the expected utility of the old, there is convergence to a unique invariant distribution with a single mass point at $\{\omega^*\}$ immediately after the initial period.

Deterministic Economy. — We now consider a deterministic economy but assume that the planner respects the participation constraint of the young as well as that of the old. In this case, Assumption 2 reduces to the standard Samuelson condition. This assumption together with the strict concavity of the utility function implies that there is a unique consumption $c^y_{\min} < e^y$, which is the lowest *stationary* consumption of the young that satisfies the participation constraint with equality, that is, $u(c^y_{\min}) + \beta u(e - c^y_{\min}) = \hat{v}$ where $\hat{v} := u(e^y) + \beta u(e^o)$. The corresponding utility of the old is $\omega_{\max} = u(e - c^y_{\min})$. Analogously to condition (5), the first-best consumption c^{y*} satisfies $u_c(c^{y*})/u_c(e-c^{y*}) = \max\{\beta/\delta, \lambda\}$ where $\lambda = u_c(e^y)/u_c(e^o)$ and the corresponding utility of the old is $\omega^* = u(e - c^{y*})$. Since $\beta/\delta > \beta > \lambda$, the participation constraint of the old is satisfied at c^{y*}. Whether the participation constraint of the young is satisfied at c^{y*} depends on the value of δ. If δ is above a critical value, then $c^{y*} > c^y_{\min}$ and the first-best consumption is sustainable. Otherwise, the first-best consumption is not sustainable.

Denote the consumption of the young at time t by c^y_t and the corresponding utility of the old by $\omega_t = u(e - c^y_t)$. Consider the maximization problem in (2) with the participation

[11] The proof of Proposition 2 is omitted because it follows from standard arguments. Nonetheless, the properties of the function $V^*(\omega)$ are mirrored in Proposition 3 and Lemma 1, both below, which do respect the participation constraints of the young.

constraints of the young given by $u(c_t^y) + \beta u(e - c_{t+1}^y) \geq \hat{v}$ for $t \geq 0$. The solution to this problem is $c_t^y = \max\{c^{y*}, c_{\min}^y\}$ for all t. For $\omega \leq \omega^*$, constraint (3) does not bind and it is optimal to set $c_t^y = \max\{c^{y*}, c_{\min}^y\}$ for all t. On the other hand, consider a case in which δ is large enough such that the first-best consumption is sustainable and $\omega \in (\omega^*, \omega_{\max})$. In this case, at $t = 0$, c_0^y must satisfy $u(e - c_0^y) \geq \omega$, which requires that $c_0^y < c^{y*}$. Clearly, it is desirable to set c_0^y such that $u(e - c_0^y) = \omega$ and $c_1^y = c^{y*}$. However, setting $c_1^y = c^{y*}$ may violate the participation constraint of the young. In such a case, c_1^y has to be chosen to satisfy $u(c_0^y) + \beta u(e - c_1^y) = \hat{v}$, which implies that $c_1^y < c^{y*}$. Repeating this argument for $t > 1$ shows that given c_t^y, the consumption of the young at time $t+1$ either satisfies $u(c_t^y) + \beta u(e - c_{t+1}^y) = \hat{v}$ or $c_{t+1}^y = c^{y*}$ if $u(c_t^y) + \beta u(e - c^{y*}) \geq \hat{v}$. It is useful to express this rule in terms of a policy function that determines the next-period value of the utility of the old, ω', as a function of the current value ω:

$$\omega' = f(\omega) := \begin{cases} \omega^* & \text{for } \omega \in [\omega_{\min}, \omega^c], \\ \frac{1}{\beta}\left(\hat{v} - u\left(e - u^{-1}(\omega)\right)\right) & \text{for } \omega \in (\omega^c, \omega_{\max}], \end{cases} \tag{7}$$

where $\omega_{\min} = u(e^o)$ and $\omega^c := u(e - u^{-1}(\hat{v} - \beta\omega^*))$. It follows from the strict concavity of the utility function that $\omega^c > \omega^*$. The function $f(\omega)$ is increasing and convex in ω as illustrated in Figure 1. The dynamic evolution of ω_t is straightforwardly derived from $f(\omega)$: for $\omega_t \in [\omega_{\min}, \omega^c]$, $\omega_{t+1} = \omega^*$ for all t; for $\omega_t \in (\omega^c, \omega_{\max}]$, ω_{t+1} declines monotonically. Since $\omega^c > \omega^*$, the process for ω_t converges to ω^*, attaining its long-run value in finite time. Intuitively, if the utility of the old is large (or equivalently, the consumption of the young is low), then the planner would like to reduce ω to ω^* (or equivalently, raise the consumption of the young to c^{y*}) as fast as possible to improve welfare. But if the consumption of the next-period young is raised too much, it will violate the participation constraint of the current young. The presence of limited enforcement means that the consumption of the young has to be raised gradually.

Denote the per-period payoff to the planner with the first-best allocation in the absence of uncertainty by $v^* := u(c^{y*}) + (\beta/\delta)\omega^*$ and the expected discounted payoff to the planner for $\omega_t \in \Omega$ by $V(\omega_t)$. The optimal solution for the deterministic case with sustainable ω^* is summarized in the following proposition.

Proposition 3. (i) If $\omega \in [\omega_{\min}, \omega^*]$, then the consumption of the young is $c_t^y = c^{y*}$ for $t \geq 0$, where $u_c(c^{y*})/u_c(e - c^{y*}) = \beta/\delta$. (ii) If $\omega \in (\omega^*, \omega_{\max}]$, then the utility of the old ω_{t+1} satisfies equation (7). There exists a finite $\hat{t}$ such that ω_t is monotonically decreasing for $t < \hat{t}$ and $\omega_t = \omega^*$ for $t \geq \hat{t}$. Likewise, c_t^y is monotonically increasing for $t < \hat{t}$ and $c_t^y = c^{y*}$ for $t \geq \hat{t}$. (iii) The value function $V\colon \Omega \to \mathbb{R}$ is equal to $V(\omega) = v^*/(1 - \delta)$

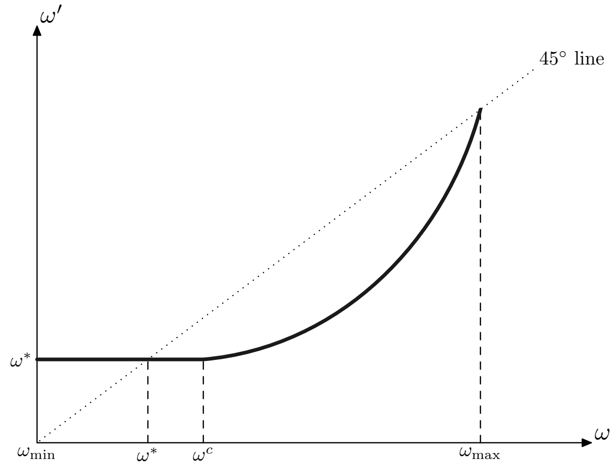

Figure 1: Policy Function in the Deterministic Case.

Note. — The solid line is the deterministic policy function $f(\omega)$. For any initial $\omega \in [\omega_{\min}, \omega_{\max})$, ω_t converges to ω^*.

for $\omega \in [\omega_{\min}, \omega^*]$ and is strictly decreasing and strictly concave for $\omega \in (\omega^*, \omega_{\max}]$ with $\lim_{\omega \to \omega_{\max}} V_\omega(\omega) = -\infty$.

The optimal solution is either stationary or converges monotonically to a stationary point within finite time with $c^y_T = c^{y*}$ for T large enough. Hence, the long-run distribution of ω is degenerate and for the case where $c^{y*} > c^y_{\min}$, it has a single mass point at $\{\omega^*\}$.

In the following sections, we show that in the case where the participation constraint of the young binds, and there is more than one endowment state, the long-run distribution of ω is non-degenerate. The benchmarks highlight that both limited enforcement of transfers and risk are necessary for the long-run distribution to be non-degenerate.

III Optimal Sustainable Intergenerational Insurance

In this section, we characterize the optimal intergenerational insurance rule under uncertainty when the planner respects the participation constraints of both the young and the old. We rule out the case in which the first-best outcome is sustainable and assume that the first-best transfers violate the participation constraint of the young in at least one state.

Assumption 4. At the first best, $u(c^{y*}(s)) + \beta \sum_{s'} \pi(s') u(e(s') - c^{y*}(s')) < \hat{v}(s)$ for at least one state $s \in \mathcal{S}$.

We reformulate the optimization problem described in Definition 2 recursively. This reformulation is possible because the endowment states are i.i.d. and all constraints are forward looking. Our characterization is similar to the promised-utility approach (see, for example, Green, 1987; Spear and Srivastava, 1987; Thomas and Worrall, 1988). For simplicity of notation, we often omit the time t indexes and use primes to denote next-period variables. At each period, the expected utility $\omega \in \Omega$ promised to the current old embodies information about the history of shocks. The problem at each period is to determine the state-contingent consumption of the young, $c^y(s)$, and the state-contingent promise of expected utility, $\omega'(s)$, made to the young for their old age.

Since $\hat{v}(s) = u(e^y(s)) + \beta\omega_{\min}$, the participation constraint of the young, constraint (1), can be rewritten as:

$$u(c^y(s)) + \beta\omega'(s) \geq u(e^y(s)) + \beta\omega_{\min} \qquad \forall\, s \in \mathcal{S}. \tag{8}$$

The participation constraint of the old is subsumed by the requirement that $c^y(s) \in \mathcal{Y}(s)$ for all s. In each period, the utility promises made to the young must be feasible:

$$\omega'(s) \leq \omega_{\max} \qquad \forall\, s \in \mathcal{S}; \tag{9}$$

and the expected utility of the current old must be at least that previously promised:

$$\sum\nolimits_s \pi(s) u(e(s) - c^y(s)) \geq \omega. \tag{10}$$

Constraint (10) is a promise-keeping constraint and is analogous to constraint (3), but is now required to hold in every period. Let $\Phi := \{\{c^y(s) \in \mathcal{Y}(s), \omega'(s) \in \Omega\}_{s\in\mathcal{S}} \mid (8), (9) \text{ and } (10)\}$ denote the constraint set. Since utility is strictly concave and Ω is an interval, Φ is convex and compact. The value function $V(\omega)$ satisfies the following functional equation:

$$V(\omega) = \max_{\{c^y(s),\omega'(s)\}_s \in \Phi} \left[\sum\nolimits_s \pi(s) \left(\tfrac{\beta}{\delta} u(e(s) - c^y(s)) + u(c^y(s)) + \delta V(\omega'(s))\right)\right]. \tag{11}$$

Denote the state vector by $x := (\omega, s)$ and the two stochastic policy functions that solve (11) by $c^y(x)$ and $f(x)$. The policy functions $c^y(x)$ and $f(x)$ are the optimal consumption of the young and the optimal promise of expected utility for their old age when the current old have a utility promise of ω and the current endowment state is s. The optimal allocation is solved recursively. First, starting from an $\omega_0 \in \Omega$ (how ω_0 is determined is discussed below), solve the maximization problem of equation (11) to obtain the policy functions $c^y(\omega_0, s)$ and $f(\omega_0, s)$. Next, depending on the endowment state realised, say s_0, input the future promised utility $f(\omega_0, s_0)$ into equation (10) and resolve the maximization problem for the next period, and so on. The compactness of Ω

and Φ guarantees the existence of the optimal allocation, and the strict concavity of $u(\cdot)$ guarantees its uniqueness.

The function $V(\omega)$ cannot be found by a standard contraction mapping argument starting from an arbitrary value function because the value function associated with the allocation in autarky also satisfies the functional equation (11). Nevertheless, a similar approach can be used to iterate the value function, starting from the first-best value function derived in Proposition 2. Following the arguments of Thomas and Worrall (1994), it can be shown that the limit of this iterative mapping is the optimal value function $V(\omega)$.[12] Proposition 2 established that the first-best value function is non-increasing, differentiable and concave, and the limit value function inherits these properties.

LEMMA 1. The value function $V\colon\Omega \to \mathbb{R}$ is non-increasing, concave and continuously differentiable in ω, where $\Omega = [\omega_{\min}, \omega_{\max}]$ and $\omega_{\min} < \omega_{\max} < \sum_s \pi(s)u(e(s))$. There is an $\omega_0 \in (\omega_{\min}, \omega^*)$ such that $V(\omega)$ is constant for $\omega \leq \omega_0$ and is strictly decreasing and strictly concave for $\omega > \omega_0$ with $V_\omega(\omega_0) = 0$ and $\lim_{\omega \to \omega_{\max}} V_\omega(\omega) = -(\beta/\delta)\nu_{\max}$, where $\nu_{\max} \in \mathbb{R}_+ \cup \{\infty\}$.

The concavity of the objective function and the convexity of the constraint set guarantee the concavity of the value function. The lower endpoint $\omega_{\min}$ of the domain of $V(\omega)$ is the autarkic value since zero transfers are feasible. The upper endpoint $\omega_{\max}$ is determined by choosing the consumption of the young and the promised utility that maximizes the expected utility of the current old subject to constraints (8) and (9). This maximization problem is itself a strictly concave programming problem and has a unique solution.[13] The latter part of Assumption 1 is sufficient to guarantee that $\omega_{\max} < \sum_s \pi(s)u(e(s))$. Differentiability follows because the constraint set satisfies a linear independence constraint qualification when $\omega \in [\omega_{\min}, \omega_{\max})$. The left-hand derivative of $V(\omega)$ evaluated at $\omega_{\max}$ is finite if $\omega_{\max}$ is part of the ergodic set and infinite otherwise. These two possibilities are discussed in more detail below.

Given Lemma 1, the planner chooses an initial promise $\omega_0 = \sup\{\omega \mid V_\omega(\omega) = 0\}$. For $\omega < \omega_0$, constraint (10) does not bind and the promise to the initial old can be increased without reducing the planner's payoff. Therefore, attention can be restricted to promises $\omega \geq \omega_0$. The initial promise ω_0 is determined as part of the optimal solution and depends, in general, on all parameter values. However, Assumption 4 implies that the first-best outcome violates one of the participation constraints in (8) and hence, $\omega_0 < \omega^*$.

[12] We use this iteration procedure to compute the optimal value function for the numerical examples considered in Sections VI and VII.

[13] Finding $\omega_{\max}$ is straightforward since the value function $V(\omega)$ does not enter into the constraint set.

Optimal Policy Functions. — We now turn to the properties of the policy functions $c^y(x)$ and $f(x)$. Given the differentiability of the value function, the first-order conditions for the programming problem in equation (11) are:

$$\frac{u_c(c^y(x))}{u_c(e(s)-c^y(x))} = \frac{\beta}{\delta}\left(\frac{1+\nu(\omega)+\eta(x)}{1+\mu(x)}\right), \tag{12}$$

$$V_\omega(f(x)) = -\tfrac{\beta}{\delta}\left(\mu(x)-\xi(x)\right), \tag{13}$$

where $\pi(s)\mu(x)$ are the multipliers associated with the participation constraints of the young (8), $\beta\pi(s)\xi(x)$ are the multipliers associated with the upper bound on the promised utility (9), $(\beta/\delta)\nu(\omega)$ is the multiplier associated with the promise-keeping constraint (10) and $(\beta/\delta)\pi(s)\eta(x)$ are the multipliers associated with the non-negativity constraints on transfers. Given the concavity of the programming problem, conditions (12) and (13) are necessary and sufficient. There is also an envelope condition:

$$V_\omega(\omega) = -\tfrac{\beta}{\delta}\nu(\omega). \tag{14}$$

Taken together, equations (13) and (14) imply the following updating property:

$$\nu(\omega') = \mu(x) - \xi(x). \tag{15}$$

Equation (15) is easily interpreted. For simplicity, suppose that the upper bound constraint on promises and the non-negativity constraint on transfers do not bind, that is, $\xi(x) = \eta(x) = 0$. From equation (12), it follows that $1+\mu(x)$ is the relative weight placed on the utility of the young and $1+\nu(\omega)$ is the relative weight placed on the utility of the old. The updating property in equation (15) shows that the relative weight placed on the utility of the old corresponds to the tightness of the participation constraint they faced when they were young.

The policy function for the future promise of expected utility is the key to understanding the evolution of the intergenerational insurance rule. It has the following properties:[14]

Lemma 2. (i) The policy function $f\colon \Omega\times\mathcal{S} \to [\omega_0, \omega_{\max}]$ is continuous and increasing in ω and strictly increasing for $f(\omega, s) \in (\omega_0, \omega_{\max})$. (ii) For at least one state r, there is a critical value $\omega^c(r) > \omega_0$ with $f(\omega, r) = \omega_0$ for $\omega \in [\omega_0, \omega^c(r)]$. (iii) For each state s, there is a unique fixed point $\omega^f(s)$ of the mapping $f(\omega, s)$ with $f(\omega, s) > \omega$ for $\omega < \omega^f(s)$ and $f(\omega, s) < \omega$ for $\omega > \omega^f(s)$. For at least one state, $\omega^f(s) > \omega_0$. (iv) If the

[14] To avoid the clumsy terminology of non-decreasing or weakly increasing, we describe a function as increasing if it is weakly increasing and highlight cases where a function is strictly increasing.

aggregate endowment is fixed, then $f(\omega, s)$ is decreasing in s and strictly decreasing for $f(\omega, s) \in (\omega_0, \omega_{\max})$.

Figure 2 depicts an example, with three endowment states, of the policy function $f(\omega, s)$. The key properties of $f(\omega, s)$ are that, for a given s, it is continuous and increasing in ω, cuts the 45° line at most once from above (see state 1 in Figure 2) and there is a state such that $f(\omega, s)$ is constant in ω for some $\omega > \omega_0$ (see states 2 and 3 in Figure 2).

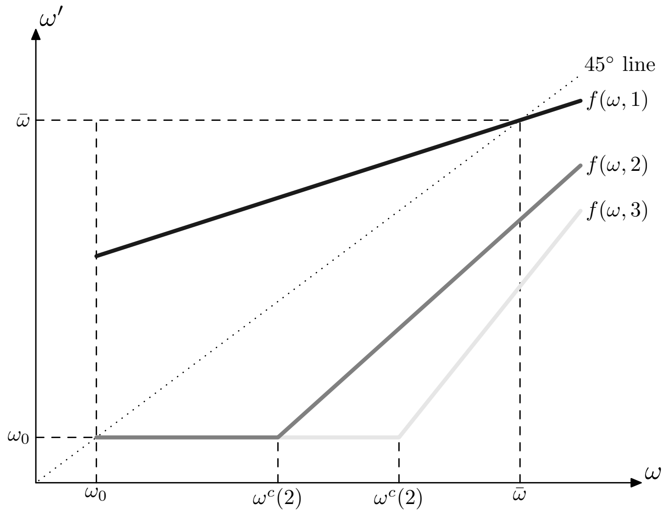

Figure 2: The Policy Function $f(\omega, s)$.

Note. — The policy function $f(\omega, s)$ represents the future promise $\omega'(s)$ for $s \in \{1, 2, 3\}$ as a function of ω. The fixed points are $\omega^f(2) = \omega^f(3) = \omega_0$ and $\omega^f(1) = \bar{\omega}$, the largest fixed point. The upward sloping parts of the policy functions are drawn as linear for illustrative purposes.

The continuity of $f(\omega, s)$ in ω follows from the strict concavity of the programming problem. Intuitively, $f(\omega, s)$ is increasing in ω. A higher promise to the current old means lower consumption for the current young. For endowment states in which the participation constraint binds, lower consumption for the current young requires a higher future promised utility as compensation. Since the value function is decreasing in ω, a higher promise lowers the planner's payoff. Thus, the planner will want to reduce the future promise whenever the participation constraint permits. In particular, if the participation constraint of the young does not bind in state s, then it follows from the first-order conditions that $f(\omega, s) = \omega_0$. That is, the promised utility is *reset* to its initial value whenever the participation constraint of the young does not bind. Once the promised utility is reset, it is as if the history of shocks is forgotten. Resetting occurs because not all participation constraints bind at ω_0, which follows from Assumption 3 that there are states in which there is no transfer at ω_0. For these states, the participation constraint of the young is strictly satisfied since $\omega_0 > \omega_{\min}$. This establishes property (*ii*) of Lemma 2. Part (*iii*) of Lemma 2 shows that the policy function $f(\omega, s)$ cuts the 45°

line once from above and part (*iv*) shows that the promised utility is monotonic in s if there is no aggregate risk. These two latter properties are discussed after the following lemma, which establishes the properties of the policy function for the consumption of the young.

LEMMA 3. (i) The policy function $c^y\colon \Omega \times \mathcal{S} \to \mathbb{R}$ is continuous and decreasing in ω and strictly decreasing for $c^y(\omega, s) < e^y(s)$ and $f(\omega, s) < \omega_{\max}$. (ii) At the fixed point $\omega^f(s)$, $c^y(\omega^f(s), s) \leq c^{y*}(s)$ and with equality for $\omega^f(s) < \omega_{\max}$. (iii) If the aggregate endowment is fixed, then $c^y(\omega, s)$ is decreasing in s.

As mentioned above, the consumption of the young is decreasing in ω. To understand part (*ii*) of Lemma 3 (and part (*iii*) of Lemma 2), suppose for simplicity that $\eta(x) = \xi(x) = 0$. From equations (13) and (14), a fixed point of $f(x)$ corresponds to a stationary point of the updating condition (15), that is, $\mu(x) = \nu(\omega)$. Substituting this condition into (12) shows that the ratio of the marginal utilities equals β/δ and hence, consumption is at the first-best level: $c^y(\omega^f(s), s) = c^{y*}(s)$. Likewise, for $f(x) > \omega$, where the next-period promise is higher than today's promise, the consumption of the young is higher than the first-best consumption; and for $f(x) < \omega$, where the next-period promise is lower than today's promise, the consumption of the young is lower than the first-best consumption. To understand why $f(x)$ cuts the 45° line from above, consider some $\omega > \omega^f(s)$ and suppose, to the contrary, that $f(x) \geq \omega$. This would imply that the consumption of the young is no lower, and that the promised utility is higher, at ω than at $\omega^f(s)$. Since the participation constraint of the young binds at ω, this would violate the participation constraint at $\omega^f(s)$. A similar argument shows that $f(x) > \omega$ for $\omega < \omega^f(s)$.[15]

The policy functions $c^y(\omega, s)$ and $f(\omega, s)$ need not be monotonic in s. However, as stated in part (*iv*) of Lemma 2 and part (*iii*) of Lemma 3, both functions are decreasing in s when the aggregate endowment is fixed. This monotonicity is intuitive because, absent aggregate risk, the ordering convention for $\lambda(s)$ implies that the autarkic utility is decreasing in s. Given that the autarkic utility is decreasing in s, if the participation constraint of the young binds in two different endowment states, then either the consumption of the young or the future promise has to be lower in the higher of the two states. The lemmas show neither the consumption of the young nor the future promise is increasing in the state at the optimum.[16] Monotonicity of $f(\omega, s)$ in s can also be used to show

[15] The argument can be extended to the case where the non-negativity and upper bound constraints bind and the complete proof of Lemma 2 is given in the Appendix.

[16] The result regarding monotonicity of the policy function $f(\omega, s)$ in s can be extended to the case in which $e^y(1) \geq e^y(2) \geq \cdots \geq e^y(S)$ and $e(S) \geq e(S-1) \geq \cdots \geq e(1)$, that is, the endowment of the young

that, absent aggregate risk, the fixed points $\omega^f(s)$ are ordered by state: $\omega^f(s) \geq \omega^f(r)$ for $s < r$, with strict inequality unless $\omega^f(s) = \omega^f(r) = \omega_0$ or $\omega^f(s) = \omega^f(r) = \omega_{\max}$.

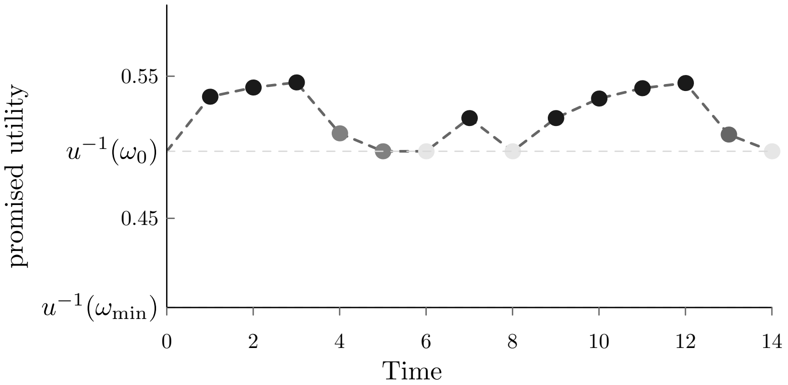

Figure 3: Sample Path of the Promised Utility.

Note. — The shade of the dots indicates the state s_t: dark gray for $s_t = 1$, mid gray for $s_t = 2$ and light gray for $s_t = 3$. Promised utility is converted to certainty-equivalent consumption $u^{-1}(\omega)$.

Figure 3 illustrates the evolution of promised utility for a given history of shocks, corresponding to the three-state example illustrated in Figure 2. The given history creates a particular sample path of the promise ω. A sample path for the history $s^T = (s_0, s_1, \ldots, s_T)$ is constructed iteratively from the policy function $f(\omega, s)$ starting with ω_0 as follows: $\omega_1 = f_1(\omega_0, s_0) := f(\omega, s_0)$, $\omega_2 = f_2(\omega_0, s^1) := f(f_1(\omega_0, s_0), s_1)$ and so on, up to $f_{T+1}(\omega, s^T) := f(f_T(\omega, s^{T-1}), s_T)$. Figure 3 illustrates two important features. First, the path is history dependent. That is, promised utility (and hence, consumption) varies both with the current endowment state and the history of shocks. For example, state 1 occurs at both $t = 7$ and $t = 12$, but the promised utility is different at the two dates. In particular, whenever state 1 occurs, the participation constraint of the young binds and a higher promised utility has to be offered to them so that they are willing to share more of their current relatively high endowment. Subsequent realizations of state 1 exacerbate the situation because the young of the next generation must deliver on past promises as well. This property is evident in Figure 3 where ω_t increases whenever state 1 repeats. Secondly, there are points in time at which promised utility resets to ω_0. In the case illustrated in Figure 2, this happens whenever state 3 occurs and sometimes when state 2 occurs. Before resetting occurs, the effect of a shock persists. However, once resetting has occurred, the history of shocks is forgotten and the subsequent sample

is weakly decreasing in s but aggregate endowment is weakly increasing in s. By continuity, monotonicity of $f(\omega, s)$ in s is also preserved provided that aggregate risk is not too large. The convergence property discussed in Section IV depends only on the monotonicity of the policy function $f(\omega, s)$ in ω and does not depend on whether it is monotonic in s.

path is identical whenever the same sequence of states occurs. That is, the sample paths between resettings are probabilistically identical.[17]

Implications for the Design of Social Security. — The results of this section provide the basis for an interpretation of the optimal sustainable intergenerational insurance as a pay-as-you-go social security or state pension system.[18] The optimum can be expressed in terms of how the pension received depends on the contribution made when young. This reinterpretation can be simply and succinctly stated with a change of variables. Let $z := u(e^y) - u(e^y - \tau)$ denote the contemporaneous utility loss of the young from making a transfer τ and let $\mathcal{Z} := [0, z_{\max}]$ be the set of feasible z, where $z_{\max} := \beta(\omega_{\max} - \omega_{\min})$. The planner can be thought of as choosing a policy function $g^z(\omega, s)$ that determines z rather than choosing the consumption of the young $c^y(\omega, s)$. The optimal policy $g^z\colon \Omega \times \mathcal{S} \to \mathcal{Z}$ inherits the properties of $c^y(\omega, s)$. In particular, it is increasing in ω, strictly increasing for $g^z(\omega, s) > 0$, and, absent aggregate risk, it is decreasing in s.

The advantage of using $g^z(\omega, s)$ as the policy function is that the future promise depends only on z and is independent of the current endowment state. Denote the future promise ω' by the function $f^z\colon \mathcal{Z} \to \Omega$ and rewrite the participation constraint of the young as $-z + \beta f^z(z) \geq \omega_{\min}$. Letting $z^c := \beta(\omega_0 - \omega_{\min})$, it follows from the participation constraint that $f^z(z) = \omega_0$ if $z \leq z^c$ and $f^z(z) = \omega_{\min} + z/\beta$ otherwise.

The functions $g^z(\omega, s)$ and $f^z(z)$ can be used to write the pension of the old as a function of z^-, the utility loss of the transfer made by the young in the previous period, and the current endowment state.[19] Denote the pension function by $h^z(z^-, s)$ where

$$h^z(z^-, s) := \begin{cases} e^y(s) - u^{-1}\left(u(e^y(s)) - g^z(\omega_0, s)\right) & \text{if } z^- \leq z^c, \\ e^y(s) - u^{-1}\left(u(e^y(s)) - g^z\left(\omega_{\min} + \frac{z^-}{\beta}, s\right)\right) & \text{if } z^- > z^c. \end{cases} \tag{16}$$

Equation (16) shows that z^- is sufficient to capture all the relevant information about the past. The properties of the pension function are described by the following corollary to Lemmas 2 and 3.

Corollary 1. (i) The pension function $h^z\colon \mathcal{Z} \times \mathcal{S} \to \mathcal{Y}(s)$ is continuous and increasing in z^- and strictly increasing for $h^z(z^-, s) > 0$. (ii) If the aggregate endowment is fixed, then $h^z(z^-, s)$ is decreasing in s and strictly decreasing for $h^z(z^-, s) > 0$.

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[17] This property is used in Section IV to establish convergence to a unique invariant distribution.

[18] In a pay-as-you-go scheme, pensions are paid from the contributions of the current young. Most state pension schemes are wholly or substantially pay-as-you-go. A funded scheme, in contrast, invests the contributions of the young in financial or other assets to be paid out during retirement.

[19] In the initial period, z^- can be set equal to any value in $\mathcal{Z}$ less than or equal to z^c.

The pension function exhibits three main features: (*i*) it depends only on the current endowment state if the utility loss when young was low, that is, when $z^{-} < z^c$; (*ii*) for each endowment state, it is increasing in the utility loss when young if the loss was high, that is, when $z^{-} > z^c$; and (*iii*) absent aggregate risk, it is decreasing in the endowment of the old.

For specific preferences, the utility loss of the young can be directly expressed in terms of their contribution. For example, consider the case of CRRA preferences where $u(c) = (c^{1-\gamma} - 1)/(1-\gamma)$ with coefficient of relative risk aversion $\gamma > 1$ and $u(c) = \log(c)$ in the limit as $\gamma \to 1$. Define

$$\zeta := 1 - u^{-1}(-z^{-}) = 1 - \left(1 + (e^y)^{1-\gamma}\left(\left(1 - \frac{\tau^{-}}{e^y}\right)^{1-\gamma} - 1\right)\right)^{\frac{1}{1-\gamma}},$$

where τ^{-}/e^y is the *contribution rate*. Note that $\zeta \in [0,1]$ is an increasing function of the contribution rate, with $\zeta = 0$ for $\tau^{-}/e^y = 0$ and $\lim_{(\tau^{-}/e^y)\to 1} \zeta = 1$. Moreover, ζ converges to τ^{-}/e^y as $\gamma \to 1$. Hence, ζ can be interpreted as a *modified* contribution rate, which is equal to the contribution rate when preferences are logarithmic. Since z and ζ are monotonically related, it follows from equation (16) that there is a corresponding function $h^\zeta(\zeta, s)$ that determines the pension transfer to the old as a function of the (modified) contribution rate paid when young and the current endowment state, with a threshold of $\zeta^c := 1 - u^{-1}(-z^c)$.[20]

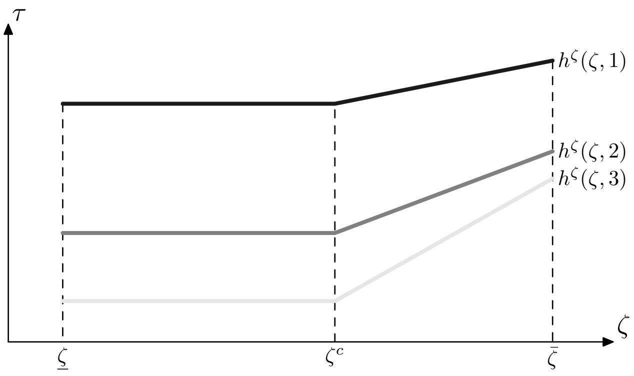

Figure 4: The pension function $h^\zeta(\zeta, s)$.

Note. — The pension functions plot the pension of the old as a function of the contribution rate ζ made when young and the current endowment state s. $\underline{\zeta}$ and $\bar{\zeta}$ are the smallest and greatest values of the contribution rates. The upward sloping parts of the pension functions are shown as linear for illustrative purposes. The case illustrated has $\underline{\zeta} > 0$ and hence, $h^\zeta(\zeta, s)$ is strictly increasing for $\zeta > \zeta^c$.

[20] It is also possible to consider alternative parameterizations of preferences. For example, with CARA preferences, the pension transfer depends on the level of contribution rather than the contribution rate.

Figure 4 illustrates the pension function $h^\zeta(\zeta, s)$ when there are three endowment states. It shows that the optimal pension scheme has both means testing and a mixture of flat-rate and contributory-related elements. If the contribution rate paid when young is low ($\zeta \leq \zeta^c$), then $h^\zeta(\zeta, s)$ is constant in ζ (the flat-rate element). If the contribution rate paid when young is high ($\zeta > \zeta^c$), then $h^\zeta(\zeta, s)$ is increasing in ζ (the contributory-rate element). Furthermore, the pension depends on the current endowment state (the means-testing element). For example, in the case of a constant aggregate endowment, as illustrated in Figure 4, the pension received by the old is inversely related to their endowment. As argued in the Introduction, all three elements are commonly found in national pension schemes. The contributory-related element is a fundamental component of the optimum when there is limited enforcement of transfers. If the first best were sustainable, then the optimum would have only flat-rate and means-testing elements. With limited enforcement, the young who contribute more than the threshold must be promised a higher pension for their old age.

Although an essential part of the optimum, the contributory-related element may be non-linear and hence, it may be impractical even when preferences are logarithmic. Therefore, in Section VII, we compare the risk and welfare properties of simpler linear alternatives with and without a threshold.

IV Convergence to the Invariant Distribution

We now consider the long-run distribution of promised utilities and show that there is a countable ergodic set and strong convergence to a unique, non-degenerate invariant distribution. The future evolution of the promise ω is a Markov chain defined for any set $A \subseteq \Omega$ by the transition function $P(\omega, A) := \Pr\{\omega_{t+1} \in A \mid \omega_t = \omega\} = \sum_s \pi(s) \mathbb{1}_A f(\omega, s)$ where $\mathbb{1}_A f(\omega, s) = 1$ if $f(\omega, s) \in A$ and zero otherwise. Since the planner sets the initial promise to ω_0 and since, from Lemma 2, $f(\omega, s)$ is increasing in ω, it follows that all subsequent promises $\omega_t \geq \omega_0$.

Recall from Section III that $f_{t+1}(\omega, s^t) := f(f_t(\omega, s^{t-1}), s_t)$ where $f_1(\omega, s) := f(\omega, s)$. The monotonicity and resetting properties of Lemma 2 imply that there is a history of finite length k such that $f_k(\omega, s^{k-1}) = \omega_0$ for any $\omega \in \Omega$. That is, it is possible to find a sequence of endowment states such that the participation constraint of the young does not bind at the end of such a sequence and hence, the promise is *reset* to ω_0. This is obvious when the policy function is ordered by the endowment state. In this case, simply pick the highest endowment state and consider the positive probability path along which this state is repeated. This gives the shortest time to reach ω_0. If the policy function is not ordered by the endowment state, then a similar procedure is to take the sequence of endowment states where s_t is chosen to minimize $f(f_t(\omega, s^{t-1}), s_t)$ at each point along the path. This may be a sequence of different endowment states but after such a sequence there is a positive probability of reaching ω_0 in finite time. An immediate consequence is that Condition **M** of Stokey and Lucas (1989, page 348) is satisfied and hence, there is convergence in the uniform metric to a unique invariant probability measure $\phi(A)$.[21]

Since there is a positive probability that the promise is reset to ω_0 in finite time, the Markov chain for the promise is *regenerative* and ω_0 is a regeneration point (see, for example, Foss et al., 2018). Moreover, the process starts at a regeneration point because the planner sets the initial promise to ω_0. Let $r_\omega := \min\{k \geq 1 \mid f_k(\omega_0, s^{k-1}) = \omega\}$ denote the first time at which the promise is equal to ω starting from ω_0. Then, r_{ω_0} is the first regeneration time, that is, the first time after the initial period at which ω_0 reoccurs. Any sample path of promises can be divided into different blocks with each block starting at a regeneration time. This can be seen in Figure 3 where there are regeneration times at dates 0, 5, 6, 8 and 14. The blocks between regeneration points are not identical but by the strong Markov property they are i.i.d. That is, at each regeneration time, the past shocks are forgotten and the future evolution of the promise is probabilistically identical. The regeneration times are also i.i.d. and the expected time to regeneration is $\varpi := \mathbb{E}_0[r_{\omega_0}]$, the same for any block. The expected time to regeneration ϖ is finite since all positive probability paths must have a sequence of endowment states that lead to ω_0.

Let $R_\omega := \Pr(r_\omega < \infty)$ be the probability of attaining the promise ω in finite time starting from ω_0. If $R_\omega > 0$, then ω is said to be *accessible* from ω_0. Since ω_0 has a positive probability mass and the set of endowment states $\mathcal{S}$ is finite and time is discrete, the associated set $E := \{\omega \mid R_\omega > 0\}$ is countable. Moreover, the set E is an equivalence class because every $\omega \in E$ is accessible from ω_0 and there is always a path from every such accessible ω back to ω_0. Therefore, E is an absorbing set, that is, $P(\omega, E) = 1$ for all $\omega \in E$, and since no proper subset of E has this property, it is *ergodic* (see, for example, Stokey and Lucas, 1989, chapter 11). Let ϖ_ω denote the expected return time to the promise ω, where $\varpi_{\omega_0} \equiv \varpi$. With ϖ finite, it follows that $R_\omega = 1$ and ϖ_ω is finite for all $\omega \in E$, that is, each promise $\omega \in E$ is positive recurrent (see, for example, Meyn and Tweedie, 2009, Theorem 10.2.2).

Define $\bar{\omega} := \max_s\{\omega^f(s)\}$ to be the largest of the unique fixed points of the S mappings $f(\omega, s)$. If $\bar{\omega} < \omega_{\max}$, then any $\omega \in (\bar{\omega}, \omega_{\max}]$ is transitory and cannot be part of the ergodic

[21] Condition **M** is satisfied because there is a $k \geq 1$ and an $\epsilon > 0$ such that the k-step transition function $P^k(\omega, \{\omega_0\}) > \epsilon$. In this case, ω_0 is an atom of the Markov chain. Açikgöz (2018), Foss et al. (2018) and Zhu (2020) use similar arguments to establish strong convergence in the case of a Aiyagari precautionary savings model with heterogeneous agents.

set and hence, $E \subseteq [\omega_0, \bar{\omega})$. Since the initial promise is ω_0, attention can be restricted to promises that are accessible from ω_0, that is, promises belonging to the ergodic set E. Hence, standard results on the convergence of positive recurrent Markov chains defined on a countable space can be applied. To state these results, let P denote the transition matrix with elements $P(\omega, \omega')$ and let $P^k(\omega, \omega')$ be the elements of the corresponding k-period transition matrix.

Proposition 4. There is pointwise convergence to a unique and non-degenerate invariant distribution $\phi = \phi P$ where for each $\omega \in E$, $\phi(\omega) = \lim_{k\to\infty} P^k(\cdot, \omega) = \varpi_\omega^{-1}$. If $\bar{\omega} < \omega_{\max}$, then $\bar{\omega} \notin E$. The invariant distribution can be found iteratively using $\phi_{t+1}(\omega') = \sum_{\omega \in E} P(\omega, \omega')\phi_t(\omega)$.

The non-degeneracy of the invariant distribution in Proposition 4 follows from Assumption 4 that the first best is not sustainable. This result contrasts with the two benchmarks considered in Section II. If transfers are enforced, or if there is no risk, then convergence is to a degenerate invariant distribution with unit mass at ω^*. If $\bar{\omega} < \omega_{\max}$, then part (ii) of Lemma 3 implies that $\bar{\omega} > \omega^*$ and the promise fluctuates above and below the first-best promise ω^* even in the long run, unlike the benchmark cases. Moreover, part (i) of Lemma 2 shows that $f(\omega, s)$ is strictly increasing in ω if $\bar{\omega} < \omega_{\max}$. Hence, $\bar{\omega}$ is not accessible from ω_0 and $\bar{\omega} \notin E$.[22] The last part of Proposition 4 shows that the invariant distribution can be computed iteratively. In particular, the invariant distribution can computed using $\phi_0(\omega) = 1$ if $\omega = \omega_0$ and zero otherwise, which is the distribution corresponding to the planner's optimal choice to set the initial promise to ω_0.[23]

A straightforward corollary of Proposition 4 is that there is a unique invariant distribution $\varphi(x)$ for $x = (\omega, s)$ where $\varphi(x) = \phi(\omega)\pi(s)$ for $\omega \in E$ and $s \in \mathcal{S}$. Therefore, corresponding to the ergodic set E is another countable ergodic set $E^x := E \times \mathcal{S}$.

V Measuring Generational Risk

We use the results of the previous two sections to examine how risk is shared across and between generations. To assess generational risk sharing we employ the conditional and mean entropy measures (see, for example, Backus, Chernov, and Zin, 2014) and the bound on the variability of the implied yields introduced by Martin and Ross (2019). The

[22] While deriving the ergodic set and invariant distribution may be quite complex, Section VI examines an example with two endowment states and shows that the invariant distribution is a transformation of a geometric distribution with a denumerable ergodic set.

[23] The convergence results hold for any initial distribution $\phi_0(A)$ even if $A \not\subseteq E$ since eventually, once regeneration occurs, all subsequent promises belong to the ergodic set.

advantage of this approach is twofold. First, it allows us to decompose risk sharing into measures of risk sharing between the young and the old and risk sharing across adjacent generations of the young. Second, it is closely tied with the implied bond yields and connects with the literature on the dominant root characterization of Pareto optimality.[24]

To proceed, write the stochastic discount factor (see, for example, Huberman, 1984; Huffman, 1986; Labadie, 1986) as follows:

$$m(x,x') := \beta \frac{u_c(e(s') - c^y(x'))}{u_c(c^y(x))} = \underbrace{\delta \left(\frac{u_c(c^y(x'))}{u_c(c^y(x))} \right)}_{m^A(x,x')} \underbrace{\left(\frac{\beta}{\delta} \frac{u_c(e(s') - c^y(x'))}{u_c(c^y(x'))} \right)}_{m^B(x,x')}, \tag{17}$$

where $x = (\omega, s)$ is the current state and $x' = (f(\omega, s), s')$ is the successor state in the following period. The stochastic discount factor in equation (17) is decomposed into two terms, $m^A(x,x')$ and $m^B(x,x')$. The first component represents risk sharing *across* two adjacent generations of the young and the second component represents risk sharing *between* the young and the old at a given date.[25] Since the planner sets the initial promise to ω_0, every state x and each successor state x' belongs to the ergodic set E^x. Let Q denote the matrix of state prices $q(x,x') := \pi(x,x')m(x,x')$, where $\pi(x,x')$ and $\varrho(x,x') := q(x,x') / \sum_{x'} q(x,x')$ are the transition and risk-neutral probabilities. Given the multiplicative decomposition in equation (17), the Perron root of Q satisfies $\rho = \rho^A \rho^B$ and the associated eigenvector $\psi(x) = \psi^A(x)\psi^B(x)$, where (ρ^A, ψ^A) and (ρ^B, ψ^B) are the Perron roots and eigenvectors of the matrices corresponding to the across and between components of the stochastic discount factor.[26]

Conditional entropy is the Kullback-Liebler divergence between the rows of the matrices of $\pi(x,x')$ and $\varrho(x,x')$ corresponding to state x. Denote the conditional entropy by $L(x) := -\sum_{x'} \pi(x,x') \log(\varrho(x,x')/\pi(x,x'))$. It is well known that $L(x)$ is non-negative

[24] Two alternative measures used to assess the divergence from first-best risk sharing are the insurance coefficient (see, for example, Kaplan and Violante, 2010) and a consumption equivalent welfare change (see, for example, Song et al., 2015). We discuss these alternatives in Part F of the Supplementary Appendix and compute their values at the invariant distribution for the numerical example considered in Section VI. It is shown that using the insurance coefficient or consumption equivalent welfare change leads to similar conclusions to those presented below.

[25] Such a decomposition can also be found in Labadie (2006). The decomposition is also similar to the multiplicative decomposition into transitory and permanent components used by Alvarez and Jermann (2005). Since endowment states are i.i.d., there is no permanent component to the stochastic discount factor in equation (17). Note that although we maintain the general notational dependence on x and x', $\pi(x,x') = \pi(s')$ since endowment states are i.i.d. and $m^B(x,x')$ depends only indirectly on x.

[26] For simplicity, and because it corresponds to our numerical procedure, we assume here that the set E^x is finite and hence, ψ is an eigenvector. It is possible to adapt the arguments to the denumerable case and to more general state spaces (see, for example, Hansen and Scheinkman, 2009; Christensen, 2017).

and provides an upper bound on the expected log excess returns of any one-period asset. Hence, $L(x)$ provides a measure of the residual risk faced by a generation born in state x. It is easily shown (see, for example, Ross, 2015) that:

$$L(x) = \log\left(\sum\nolimits_{x'} \frac{\pi(x,x')}{\psi(x')}\right) - \sum\nolimits_{x'} \pi(x,x') \log\left(\frac{1}{\psi(x')}\right). \tag{18}$$

Conditional entropy measures for the across and between components of the stochastic discount factor, say, $L^A(x)$ and $L^B(x)$, are obtained by replacing ψ with ψ^A and ψ^B in (18). Note that $L(x)$ is not necessarily equal to the sum of $L^A(x)$ and $L^B(x)$ since $m^A(x,x')$ and $m^B(x,x')$ may be correlated.[27] Mean entropy, $\bar{L} := \sum_x \varphi(x) L(x)$, measures the expected conditional entropy at the invariant distribution and the corresponding mean entropies for the across and between components of risk are obtained by replacing $L(x)$ with $L^A(x)$ and $L^B(x)$ in this expectation.

Conditional entropy can easily be adapted to provide an upper bound on the expected log excess returns of any k-period asset. Conditional entropy over k periods, $L^k(x)$, is found by using the k-period transition values of $\pi^k(x,x')$ and $\varrho^k(x,x')$. Similarly, the associated mean entropy per-period, $\bar{L}^k/k$, provides a measure of long-run risk and how it is spread over time.

There is a close connection between entropy and implied yields. The continuously compounded return on a k-period bond, conditional on state x, is $y^k(x) := -(1/k)\log(p^k(x))$, where $p^k(x)$ is the bond's price.[28] The mean entropy per period satisfies:

$$\frac{\bar{L}^k}{k} = y^\infty - \bar{y}^k, \tag{19}$$

where $\bar{y}^k := \sum_x \varphi(x) y^k(x)$ is the average yield and $y^\infty := \lim_{k\to\infty} y^k(x)$ is the yield on a long bond.[29] Since mean entropy is non-negative, the average yield cannot be greater than the long-run yield. Martin and Ross (2019) show that $|y^k(x) - y^\infty| \le (1/k)\Upsilon$, where $\Upsilon := \log(\psi_{\max}/\psi_{\min})$ and $\psi_{\max}$ and $\psi_{\min}$ are the maximum and minimum values of the

[27] Letting $o(x) = \operatorname{cov}(m^A(x,x'), m^B(x,x'))/(\mathbb{E}_{x'}[m^A(x,x')] \cdot \mathbb{E}_{x'}[m^B(x,x')])$, it can be shown that $L(x) = L^A(x) + L^B(x) + \log(1+o(x))$ where the expectations and covariance are taken over x' conditional on x. Hence, $L(x) \lesseqgtr L^A(x) + L^B(x)$ as $o(x) \lesseqgtr 0$.

[28] The bond's price can be defined recursively by $p^k(x) := \sum_{x'} q(x,x') p^{k-1}(x')$ where $p^0(x) \equiv 1$.

[29] It is shown below that y^∞ is independent of the current state. Part C of the Supplementary Appendix provides details of the derivation of equation (19).

eigenvector ψ. The deviation of the yield from its mean satisfies:

$$\Delta y^k(x) := y^k(x) - \bar{y}^k \in \left[\frac{\bar{L}^k - \Upsilon}{k}, \frac{\bar{L}^k + \Upsilon}{k}\right].$$

Since $y^k(x)$ cannot be greater than the average yield for all x, it follows that $\bar{L}^k \leq \Upsilon$. While entropy is a measure of the variability in the eigenvector ψ, which in turn depends on the variability of the stochastic discount factor, the term Υ quantifies the range of the variability in ψ. It provides an upper bound on mean entropy and can be thought of as an upper bound on the residual risk.

First-Best Benchmark. — In the first best, the promised utility is constant and only the endowment state s matters. For simplicity, suppose that the non-negativity constraint on transfers does not bind. In this case, $m^{B*}(s, s') = 1$ and $m^*(s, s') = m^{A*}(s, s')$. This is akin to a representative agent model with consumption variability across time due to aggregate risk. Since the corresponding matrix of state prices has rank one, the Perron root is equal to the trace of the matrix Q and hence, $\rho^* = \rho^{A*} = \delta$ and $\rho^{B*} = 1$. The elements of the corresponding eigenvector are $\psi^*(s) = 1/u_c(c^{y*}(s))$, which depend only on the endowment state s. Since $c^{y*}(s)$ is increasing in the aggregate endowment, $\psi^*_{\max}$ corresponds to an endowment state with the highest aggregate endowment and $\psi^*_{\min}$ to an endowment state with the lowest aggregate endowment. Hence,

$$L^{k*}(s) = \log\left(\sum\nolimits_s \frac{\pi(s)}{\psi^*(s)}\right) - \sum\nolimits_s \pi(s) \log\left(\frac{1}{\psi^*(s)}\right) \;\; \forall s,\, k; \;\text{ and }\; \Upsilon^* = \log\left(\frac{\psi^*_{\max}}{\psi^*_{\min}}\right).$$

Entropy in the first best is independent of the endowment state s and time horizon k because the endowment shocks are transitory. Risk is perfectly shared between generations and there is no differential in risk shared across endowment states or time. If there is no aggregate risk, then the first-best consumption is independent of the endowment state, $m^{A*}(s, s') = \delta$ and the eigenvector ψ can be normalized to the unit vector. In this case, $L^{k*}(s) = \Upsilon^* = 0$ and $y^{k*}(s) = -\log(\delta)$ for all s and k.

Limited Enforcement. — If only partial risk sharing can be sustained, then $L^k(x)$ and $y^k(x)$ depend on the time horizon k and the state x, even in the absence of aggregate risk. In general, this dependence might be quite complex. Therefore, the following two sections explore an example with two endowment states and derive some additional theoretical and numerical results. Nevertheless, there are some properties that hold in general.

PROPOSITION 5. In the optimal sustainable intergenerational insurance: (i) The deviation $\Delta y^k(\omega, s)$ is increasing in ω for each k and s. (ii) The yield on a long bond satisfies $\lim_{k\to\infty} y^k(x) = -\log(\rho)$ for each x. (iii) If $\bar{\omega} < \omega_{\max}$, then $\rho \leq \delta$, with equality if the non-negativity constraint on transfers does not bind. (iv) Absent aggregate risk, if $\bar{\omega} < \omega_{\max}$, then $y^1(\bar{\omega}, 1) > -\log(\delta) > y^1(\omega_0, S)$.

To understand Proposition 5, recall that the consumption of the young is decreasing in ω, while the future promise is increasing in ω. Therefore, the stochastic discount factor $m((\omega, s), (f(\omega, s), s'))$ is decreasing in ω.[30] Hence, taking expectations, the price of the one-period bond decreases with ω. Equivalently, the one-period yield, $y^1(\omega, s)$, increases with ω. Thus, an agent born into a generation where the promise is higher faces higher one-period yields. Part (*i*) shows this is true for bonds of any maturity. Part (*ii*) is a standard result that all yields converge in the long run (see, for example, Martin and Ross, 2019). That is, the optimal exposure to the risk from a shock vanishes in the very long run. Part (*iii*) shows that if the upper bound on promises and the non-negativity constraint on transfers do not bind, then the long-run yield is the same as in the first best and is determined by the planner's discount factor. In particular, $\rho = \rho^A = \delta$ and $\rho^B = 1$. Finally, part (*iv*) shows that the yield is high when the young have a high endowment and the promise is large.

VI Two Endowment States

Finding the optimal sustainable intergenerational insurance is complex because it involves solving the functional equation (11). In this section, we present an example with two endowment states that can be solved using a shooting algorithm without the need to compute the value function $V(\omega)$.[31] For this case, a full characterization of the optimal dynamics of consumption and the invariant distribution of promises is provided together with solutions for the generational risk measures outlined in the previous section.

Assumptions 1-4 are maintained but we concentrate on a case with CRRA utility: $u(c) = (c^{1-\gamma} - 1)/(1 - \gamma)$ with $u(c) = \log(c)$ in the limit as $\gamma \to 1$. The two endowment states $s \in \{1, 2\}$ occur with probabilities π and $1 - \pi$. There is no aggregate risk and the aggregate endowment is normalized to unity. The endowments of the young are $e^y(1) = \kappa + \sigma$ and $e^y(2) = \kappa - \sigma\pi/(1 - \pi)$, where $\kappa \in (1/2, 1)$ and $\sigma > 0$. That is, the young are relatively rich in state 1 and relatively poor in state 2. An increase in σ is a mean-preserving spread of risk.

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[30] If consumption is ordered by endowment state (for example, if aggregate endowment is constant), then $m((\omega, s), (f(\omega, s), s'))$ is decreasing in s'. However, the dependence on s is not clear cut because a higher endowment state (lower endowment of the young) means a lower consumption of the young but also a lower future promise, leading to a lower consumption of the old in the subsequent period.

[31] See Part D of the Supplementary Appendix for details of the shooting algorithm.

By Assumptions 3 and 4, the promised utility satisfies $f(\omega_0,1) > f(\omega_0,2) = \omega_0$. From Lemma 2, $f(\omega,1) > f(\omega,2)$ and the largest fixed point $\bar{\omega} = \omega^f(1)$ is unique. We make two additional assumptions. First, that $\bar{\omega} < \omega_{\max}$: that is, the upper bound constraint (9) never binds. Second, that $f(\omega,2) = \omega_0$ for $\omega \in [\omega_0, \bar{\omega}]$: that is, the participation constraint of the poor young never binds. In this case, the multiplier $\mu(\omega,2) = 0$ for $\omega \in [\omega_0, \bar{\omega}]$ and the promised utility is optimally reset to ω_0 whenever state 2 occurs. When the young are rich, the promised utility increases, approaching $\bar{\omega}$ if state 1 is repeated infinitely often. Consequently, the history of states is forgotten whenever state 2 occurs and the future promise depends only on the number of consecutive state 1s in the most recent history. There is a non-empty set of parameter values that satisfies these two additional assumptions. For example, they are valid for the following parameter values.

Example 1. $\delta = \beta = \exp(-1/75)$, $\gamma = 1$, $\pi = 1/2$, $\kappa = 3/5$, and $\sigma = 1/10$.

Example 1 is our canonical example and all figures in this section relate to this case.[32]

Using the two additional assumptions, we proceed in three steps. First, Proposition 6 establishes the properties of the optimal consumption and the generational risk measures conditional on the current state. Second, Proposition 7 derives the invariant distribution and the moment conditions for consumption at the invariant distribution. Finally, we consider how the implied yields and mean entropy per period depend on the time horizon.

Conditional Risk. — The following proposition shows how consumption and the generational risk measures depend on the current state.

Proposition 6. Suppose $f(\omega,2) = \omega_0$ for $\omega \in [\omega_0, \bar{\omega}]$ and $\bar{\omega} < \omega_{\max}$, then: (i) The policy function $c^y(\omega,s)$ is decreasing in ω with $c^y(\omega,1) > c^y(\omega,2)$ and $c^y(\bar{\omega},1) = c^{y*}(1) = c^{y*}(2) = c^y(\omega_0,2)$. (ii) The Perron root of Q is $\rho = \rho^A = \delta$. The corresponding eigenvectors satisfy $\psi(\omega,s) = \psi^A(\omega,s)\psi^B(\omega,s)$ for $s = 1,2$ where:

$$\psi^A(\omega,1) = c^y(\omega,1)^\gamma;\ \ \psi^B(\omega,1) = \upsilon(\omega)^{-1};\ \ \psi^A(\omega,2) = c^y(\omega,2)^\gamma;\ \ \psi^B(\omega,2) = 1;$$

and $\upsilon(\omega) := 1 + \mu(\omega,1) > 1$ is increasing in ω. The upper bound on mean entropy is $\Upsilon = \Upsilon^B = \log(\upsilon(\bar{\omega}))$. (iii) Conditional entropy in state 2 is independent of ω:

$$L^A(\omega,2) = \log\left(\pi c^y(\omega_0,1)^\gamma + (1-\pi)c^{y*}(2)^\gamma\right) - \gamma\left(\pi \log\left(c^y(\omega_0,1)\right) + (1-\pi)\log\left(c^{y*}(2)\right)\right),$$
$$L^B(\omega,2) = \log\left(1 + \pi(\upsilon(\omega_0) - 1)\right) - \pi \log\left(\upsilon(\omega_0)\right),$$

[32] It can be checked that this example satisfies Assumptions 2-4. The value of δ in Example 1 corresponds to a long-run interest rate of 1⅓%.

where $c^y(\omega_0, 1) = (\upsilon(\omega_0))^{1/\gamma}/((\beta/\delta)^{1/\gamma} + \upsilon(\omega_0)^{1/\gamma})$ and $c^{y*}(2) = 1/(1 + (\beta/\delta)^{1/\gamma})$.

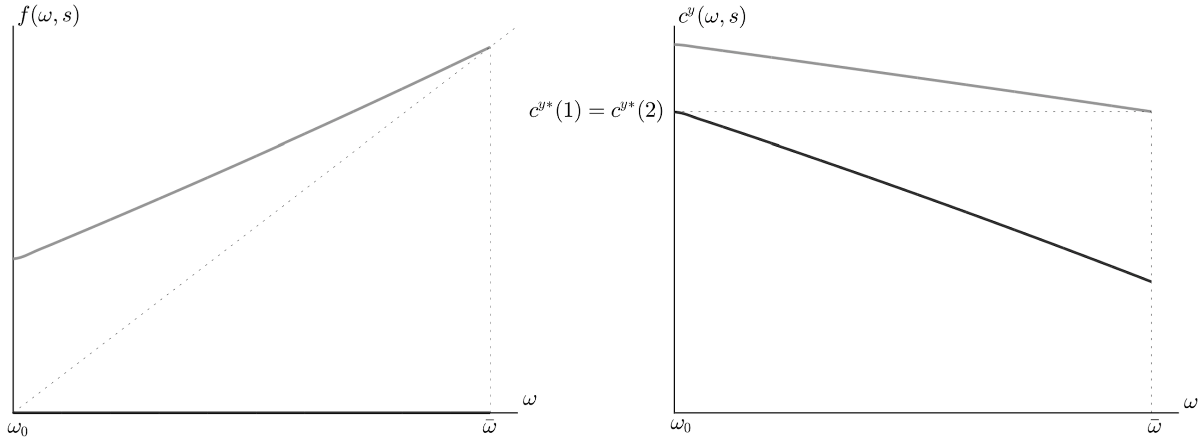


Figure 5: Panel A – Promised Utility. Panel B – Young Consumption.

Note. — In Panel A, the light gray line is $f(\omega, 1)$ and the dark gray line is $f(\omega, 2)$. The dotted line is the 45° line. In Panel B, the light gray line is $c^y(\omega, 1)$ and the dark gray line is $c^y(\omega, 2)$.

Figure 5 plots the future promised utility and the consumption of the young as a function of the current promise ω for each endowment state. The policy function $f(\omega, 1)$ is monotonically increasing in ω with a fixed point at $\bar{\omega}$, while the policy function $f(\omega, 2) = \omega_0$ for $\omega \leq \bar{\omega}$. As stated in part (i) of Proposition 6, $c^y(\omega, s)$ is monotonically decreasing in ω and ordered by the state: $c^y(\omega, 1) > c^y(\omega, 2)$. Since $\bar{\omega}$ and ω_0 are fixed points of $f(\omega, 1)$ and $f(\omega, 2)$, it follows from Lemma 3 that $c^y(\bar{\omega}, 1) = c^{y*}(1)$ and $c^y(\omega_0, 2) = c^{y*}(2)$. With no aggregate risk, the first-best consumption is independent of the state, that is, $c^{y*}(1) = c^{y*}(2)$, and hence, $c^y(\bar{\omega}, 1) = c^y(\omega_0, 2)$. In this case, for $\omega \in (\omega_0, \bar{\omega})$, the transfer is lower than the first-best transfer in state 1 and higher than the first-best transfer in state 2.

Panel A of Figure 6 plots the eigenvector $\psi(\omega, s)$ and its components $\psi^A(\omega, s)$ and $\psi^B(\omega, s)$ for each s. With $\gamma = 1$, the components $\psi^A(\omega, 1)$ and $\psi^A(\omega, 2)$ correspond exactly to consumption $c^y(\omega, 1)$ and $c^y(\omega, 2)$ illustrated in Panel B of Figure 5. Since $\upsilon(\omega) > 1$ and is increasing in ω, $\psi^B(\omega, 1) < \psi^B(\omega, 2) = 1$ and is decreasing in ω. Equally, $\psi(\omega, 1) < \psi^A(\omega, 1)$ and $\psi(\omega, 2) = \psi^A(\omega, 2)$. Moreover, $\psi_{\max} = \psi(\omega_0, 2) = c^{y*}(2)$ and $\psi_{\min} = \psi(\bar{\omega}, 1) = c^{y*}(1)/\upsilon(\bar{\omega})$. Since $c^{y*}(1) = c^{y*}(2)$, the upper bound on mean entropy is $\Upsilon = \log(\psi_{\max}/\psi_{\min}) = \log(\upsilon(\bar{\omega}))$. That is, the bound Υ depends only on the tightness of the participation constraint of the young in state 1 at $\bar{\omega}$. Furthermore, it can be derived analytically from the primitives of the model, without the need for numerical approximation, by using the promise-keeping constraint and the participation constraint of the young, both of which bind in state $(\bar{\omega}, 1)$. Likewise, $\psi^B_{\max} = 1$ and $\psi^B_{\min} = 1/\upsilon(\bar{\omega})$

and hence, $\Upsilon = \Upsilon^B$. That is, with no aggregate risk, the upper bound on mean entropy is determined by the upper bound on the between component of risk.[33]

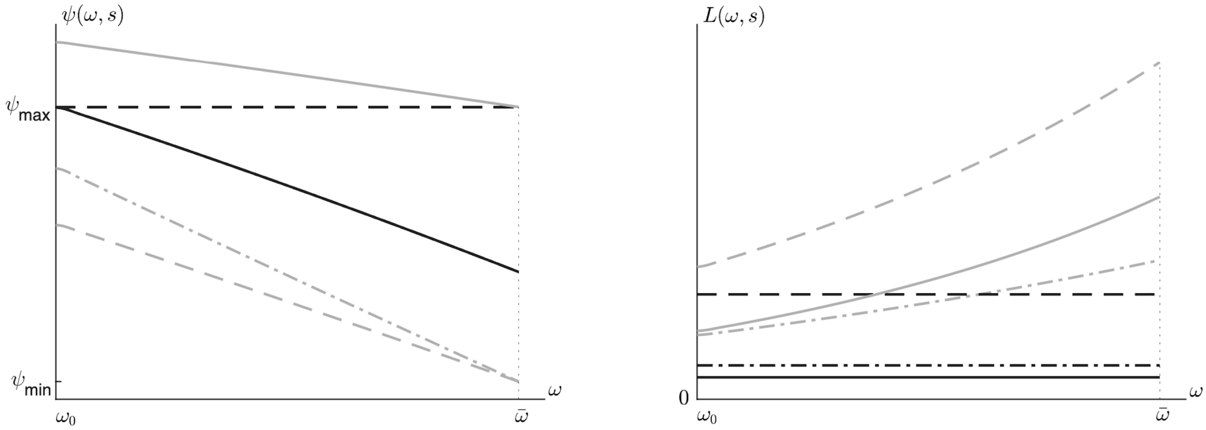


Figure 6: Panel A – Eigenvectors. Panel B – Conditional Entropy.

Note. — In Panel A, the light gray solid line is $\psi^A(\omega, 1)$ and the dark gray solid line is $\psi^A(\omega, 2)$. The light gray dashed line is $\psi^B(\omega, 1)$ and the dark gray dashed line is $\psi^B(\omega, 2)$. The light gray dot-dashed line is $\psi(\omega, 1)$. Note that $\psi(\omega, 2)$ coincides with $\psi^A(\omega, 2)$. For illustrative purposes, $\psi^B(\omega, 1)$ and $\psi(\omega, 2)$ are normalized by multiplying by $1/2$. In Panel B, the light gray solid line is $L^A(\omega, 1)$ and the dark gray solid line is $L^A(\omega, 2)$. The light gray dashed line is $L^B(\omega, 1)$ and the dark gray dashed line is $L^B(\omega, 2)$. The light gray dot-dashed line is $L(\omega, 1)$ and the dark gray dot-dashed line is $L(\omega, 2)$.

While $\Upsilon = \log(\upsilon(\bar{\omega}))$ provides an upper bound, the term $\log(\upsilon(\omega))$ provides a measure of the deviation of the between component of the stochastic discount factor from its first-best level for a given promise ω. To see this, recall that the between component depends on the ratio of marginal utilities of the young and the old at a given date. With some abuse of notation, write this as $m^B(\omega, s)$. Assuming that the non-negativity constraints on transfers do not bind, $\log(m^B(\omega, s))$ measures the deviation of $m^B(\omega, s)$ from the first best for a given (ω, s). It is easily checked that $m^B(\omega, s)$ is decreasing in ω, $m^B(\omega, 1) \geq 1 \geq m^B(\omega, 2)$ with one of the inequalities holding strictly, $m^B(\omega_0, 1) > 1 > m^B(\bar{\omega}, 2)$ and $m^B(\bar{\omega}, 1) = 1 = m^B(\omega_0, 2)$. As ω increases, the deviation of $m^B(\omega, s)$ from the first best decreases in state 1 and increases in state 2. Using the first-order condition (12), $\log(\upsilon(\omega)) = \log(m^B(\omega, 1)) - \log(m^B(\omega, 2)) > 0$. Since $\log(\upsilon(\omega))$ increases in ω, an agent born into a generation where the promise ω is higher bears more of the between component of risk.

Panel B of Figure 6 plots the conditional entropy $L(\omega, s)$ and its components $L^A(\omega, s)$ and $L^B(\omega, s)$ for each state s. These measures quantify the conditional risk faced by a generation born into state x. As shown in Part *(iii)* of Proposition 6, the entropy

[33] The exact formula for Υ is given in the proof of Proposition 6 in the Appendix and its comparative static properties are described in Part E of the Supplementary Appendix. If $e_1 > e_2$, then $\Upsilon = \Upsilon^B + \Upsilon^*$, where Υ^* is upper bound on mean entropy corresponding to the first-best allocation.

measures in state 2 are independent of ω because the future promise is always reset to ω_0 irrespective of the current promise. In state 1, the entropy measures are increasing in ω because the variability of old-age consumption increases with ω. In this example, $L^B(\omega, s) > L^A(\omega, s)$ and the between component of risk dominates, consistent with the result that $\Upsilon = \Upsilon^B$. However, since $c^y(\omega, 1) > c^y(\omega, 2)$, the across and the between components of risk are negatively correlated and hence, $L(\omega, s) < L^A(\omega, s) + L^B(\omega, s)$.[34]

Invariant Distribution. — To simplify notation, let $\omega_0^{(n)}$ denote the promised utility after n consecutive state 1s, starting from an initial promise $\omega_0^{(0)} \equiv \omega_0$.[35]

PROPOSITION 7. Suppose $f(\omega, 2) = \omega_0$ for $\omega \in [\omega_0, \bar{\omega}]$ and $\bar{\omega} < \omega_{\max}$, then the ergodic set is $E = \{(\omega_0^{(n)})_{n\geq 0}\}$ and at the invariant distribution: (i) The expected time to regeneration is $\varpi = (1-\pi)\sum_{j=1}^{\infty} j\pi^{j-1} = 1/(1-\pi)$ and the probability mass function is $\phi(\omega_0^{(n)}) = \pi^n \varpi^{-1}$ for $n = 0, 1, \ldots, \infty$. (ii) Mean entropy is given by:

$$\bar{L} = \sum\nolimits_{n=0}^{\infty} (1-\pi)\pi^{n+1} L(\omega_0^{(n)}, 1) + (1-\pi)L(2). \tag{20}$$

(iii) The logarithm of the ratio of marginal utilities is heteroskedastic with the endowment: $\mathrm{var}(\log(m^B(\omega, 2))) > \mathrm{var}(\log(m^B(\omega, 1)))$. (iv) The auto-covariance of the promised utility over two adjacent periods is positive: $\mathrm{cov}(\omega_t, \omega_{t+1}) > 0$. (v) The auto-covariance of the consumption of the young for two adjacent generations conditional on the endowment state is non-negative with $\mathrm{cov}(c_t^y, c_{t+1}^y \mid s_t = 1) > 0$ and $\mathrm{cov}(c_t^y, c_{t+1}^y \mid s_t = 2) = 0$.

Part (i) of Proposition 7 follows directly from the resetting property. With $\bar{\omega} < \omega_{\max}$, and since the promised utility is reset to ω_0 whenever state 2 occurs, the invariant distribution of ω depends only on the number of consecutive state 1s. Since the number of consecutive state 1s is geometrically distributed, so too is the invariant distribution of ω with an appropriate change of variables. It follows from Proposition 4 that the invariant distribution has a probability mass of $\phi(\omega_0) = \varpi^{-1}$ at ω_0 and has no probability mass at $\bar{\omega}$. The invariant distribution of x is also easily calculated from ϕ because $\varphi(\omega, s) = \pi(s)\phi(\omega)$. Using the invariant distribution $\varphi(\omega, s)$, mean entropy is given by equation (20).

Part (iii) of Proposition 7 shows that the consumption allocation is heteroskedastic. For Example 1, the variance of the consumption of the young is lower when their endowment is higher. The results in parts (iv) and (v) of Proposition 7 are illustrated in Figure 7, which plots the joint distribution of the promised utility (Panel A) and the joint

[34] The covariance of $m^A(x, x')$ and $m^B(x, x')$ is negative for this two-state example.

[35] Formally, $\omega_0^{(n)} = f_1^{(n)}(\omega_0)$, $f_1(\omega) := f(\omega, 1)$ and $f_1^{(n)}$ is the n-fold composition of f_1 with $f_1^{(0)}(\omega) = \omega$. Since $\bar{\omega}$ is the fixed point of the policy function $f(\omega, 1)$, $\lim_{n\to\infty} \omega_0^{(n)} = \bar{\omega}$.

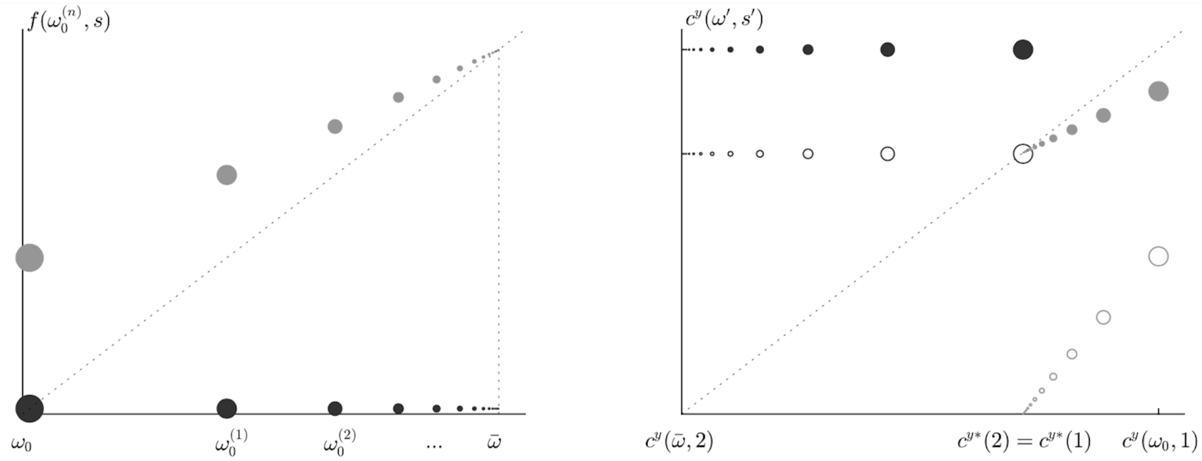


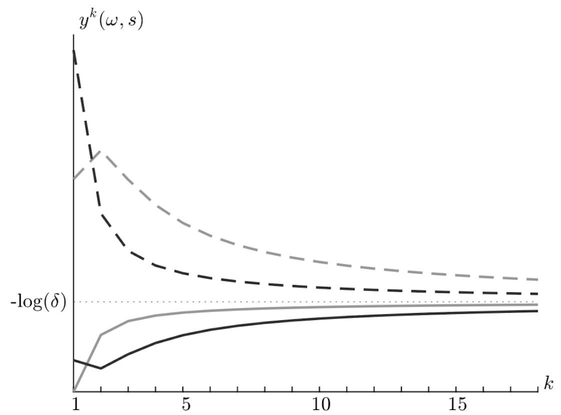
Figure 7: Panel A – Joint Distribution of ω. Panel B – Joint Distribution of $c^y(\omega, s)$.
Note. — In both panels, the size of the dot represents the frequency of occurrence. Light gray dots correspond to state 1 and dark gray dots to state 2 in the current period. The dashed line is the 45° line. In Panel B, a solid dot indicates that the endowment state is the same in both periods and an open dot means transition to the other endowment state.

distribution of the consumption of the young for two adjacent generations (Panel B). It follows that $\text{cov}(\omega_t, \omega_{t+1}) > 0$ because $f(\omega, s)$ is increasing in the current promise.[36] The properties of the auto-covariance of promised utility are reflected in the auto-covariance of consumption. Conditional on state 2, the auto-covariance between the consumption of the young of two adjacent generations is zero because the promise and hence, next-period consumption, is always reset whenever state 2 occurs. On the other hand, conditional on state 1, the expected consumption of the young next period is increasing in the consumption of the current young.[37] The comparison to a model of limited commitment with infinitely-lived agents is discussed in Section VIII, but it is worth noting that in an equivalent two-state case with two infinitely-lived agents, the conditional and unconditional auto-covariance of consumption across adjacent periods is zero.

Horizon Dependence. — Panel A of Figure 8 plots the yield curves $y^k(\omega_0, s)$ and $y^k(\bar{\omega}, s)$ for each s. Part (*iv*) of Proposition 5 demonstrated that $y^k(\bar{\omega}, 1) > -\log(\delta) > y^k(\omega_0, 2)$ for $k = 1$. The figure shows that the same result holds for every k and that $y^k(\bar{\omega}, 2) >$

[36] At the invariant distribution, $\text{cov}(\omega_t, \omega_{t+k})$ depends only on the time horizon k and is positive. As $k \to \infty$, the conditional expectation of ω_{t+k} converges to the mean of the invariant distribution, which is constant and independent of k. Hence, $\lim_{k\to\infty} \text{cov}(\omega_t, \omega_{t+k}) = 0$.

[37] Similarly, absent aggregate risk, there is a positive relationship for the consumption of the old: $\text{cov}(c_t^o, c_{t+1}^o \mid s_t = 1) > 0$. The unconditional auto-covariance of consumption, $\text{cov}(c_t^y, c_{t+1}^y)$, is typically negative (see Part F of the Supplementary Appendix). This is because consumption is high in the period after resetting but consumption is generally low in state 2. By the law of total covariance, $\text{cov}(c_t^y, c_{t+1}^y) = \mathbb{E}_s[\text{cov}(c_t^y, c_{t+1}^y \mid s_t)] + \text{cov}(\mathbb{E}_\varphi[c_t^y \mid s_t], \mathbb{E}_\varphi[c_{t+1}^y \mid s_t])$. The first term is positive by part (v) of Proposition 7, but the second term is negative because $\mathbb{E}_\varphi[c_t^y \mid s_t = 1] > \mathbb{E}_\varphi[c_t^y \mid s_t = 2]$ and $\mathbb{E}_\varphi[c_{t+1}^y \mid s_t = 1] < \mathbb{E}_\varphi[c_{t+1}^y \mid s_t = 2]$.



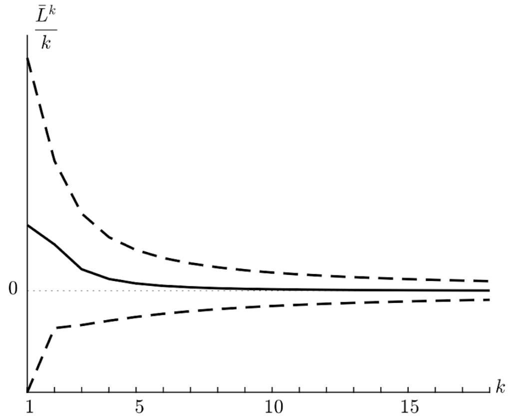


Figure 8: Panel A – Yield Curves. Panel B – Risk Measures.

Note. — In Panel A, the light gray solid line is $y^k(\omega_0, 1)$, the light gray dotted line is $y^k(\bar{\omega}, 1)$, the dark gray solid line is $y^k(\omega_0, 2)$, and the dark gray dotted line is $y^k(\bar{\omega}, 2)$. In Panel B, the solid line is $\bar{L}^k/k$ and the dashed lines are $(\bar{L}^k - \Upsilon)/k$ and $(\bar{L}^k + \Upsilon)/k$.

$-\log(\delta) > y^k(\omega_0, 1)$. Moreover, all yields converge to the long-run yield $y^\infty = -\log(\delta)$ as $k \to \infty$. Panel B of Figure 8 plots the mean entropy per period, $\bar{L}^k/k$, together with the associated bounds $(\bar{L}^k \pm \Upsilon)/k$. As shown in equation (19), the mean entropy per period equals the deviation of the average yield from the long-run yield. The dashed lines provide the bounds of $\Delta y^k(x)$ and give an indication of the maximum variability of the possible yield curves. The mean entropy per period and the associated bounds converge to zero with the horizon k showing that the influence of past shocks dies out over time. Similarly, it can be checked that the autocorrelation between the consumption of the young for generations born at time t and time $t+k$ is monotonically declining in k and tends to zero with the horizon, implying a low persistence in consumption between generations born far apart.

VII Social Security Policy Alternatives

As noted in Section III, with CRRA preferences and a coefficient of relative risk aversion greater than or equal to one, the optimal pension transfer to the old is given by a function of the modified contribution rate paid when young and the current endowment state. This function exhibits several features of observed pay-as-you-go state pension systems, with both means testing and a mixture of flat-rate and contributory-related elements. However, the optimal pension scheme may be impractical if the contributory-related element is non-linear. Therefore, in this section, we consider some simpler linear alternatives with and without a threshold and compare their welfare and risk properties using the example of Section VI.[38]

We examine three alternatives. The first is an Approximated Rule that matches the optimal function $h^{\varsigma}(\zeta, s)$ except that it imposes linearity on the contributory-related element. This rule can be described by the following *ramp function*:

$$\tilde{h}^{\varsigma}(\zeta, s) = \begin{cases} a(s) & \text{if } \zeta \leq \zeta^c, \\ a(s) + b(s)(\zeta - \zeta^c) & \text{if } \zeta > \zeta^c, \end{cases}$$

where ζ^c is the optimal threshold for the contribution rate and $a(s)$ and $b(s)$ are parameters contingent on the endowment state. The second is a Flat-rate Rule without the contributory-related element ($b(s) = 0$) and the third is a Contributory-related Rule without a threshold ($\zeta^c = 0$). All three alternatives involve means testing.

Table 1: Social Security Alternatives.

Social Security Rules	Parameter Values	Welfare Loss	Mean Entropy
Optimum	—	1.536	14.72
Approximated Rule	$a(1)=0.161$, $b(1)=0.407$ $a(2)=0.000$, $b(2)=0.935$	1.539	14.97
Flat-rate Rule	$a(1)=0.158$, $b(1)=0.000$ $a(2)=0.050$, $b(2)=0.000$	1.727	41.93
Contributory-related Rule	$a(1)=0.167$, $b(1)=0.000$ $a(2)=0.000$, $b(2)=0.243$	1.603	27.02

Note. — At the first best, $a^*(1) = 0.2$, $a^*(2) = b^*(1) = b^*(2) = 0$, the steady-state welfare is $-2\log(2)$ and the mean entropy is zero. The welfare loss is percentage loss measured relative to the first best. The mean entropy is in basis points. Both the Optimum and the Approximated Rule have the same threshold, $\zeta^c = 0.191$.

The parameters of the Approximated Rule are chosen by minimizing a constrained sum of squared errors between the ramp function $\tilde{h}^{\varsigma}(\zeta, s)$ and the optimal function $h^{\varsigma}(\zeta, s)$ for the given optimal threshold ζ^c.[39] The parameters for the Flat-rate Rule and the Contributory-related Rule are chosen to maximize the steady-state expected discounted utility subject to the participation constraints. Table 1 lists parameter values of each alternative together with the welfare loss relative to the first best and the corresponding

[38] Farhi and Werning (2013) provide a similar comparison of income tax policies. They compare a non-optimized age-dependent linear tax, which they show is close to optimal, with a simpler and numerically tractable age-independent tax for which the optimum can be computed.

[39] This procedure avoids the numerical complexity of optimizing over a non-differentiable function. It is possible that the Approximated Rule may violate the participation constraints. However, at least in this example, any violations of the participation constraints are numerically negligible.

mean entropy.[40] The welfare loss and the mean entropy of the optimal solution are also reported for comparison. Figure 9 plots a sample path of the contribution rates for the three alternatives (Panel A) and the corresponding conditional entropy along the same sample path (Panel B).

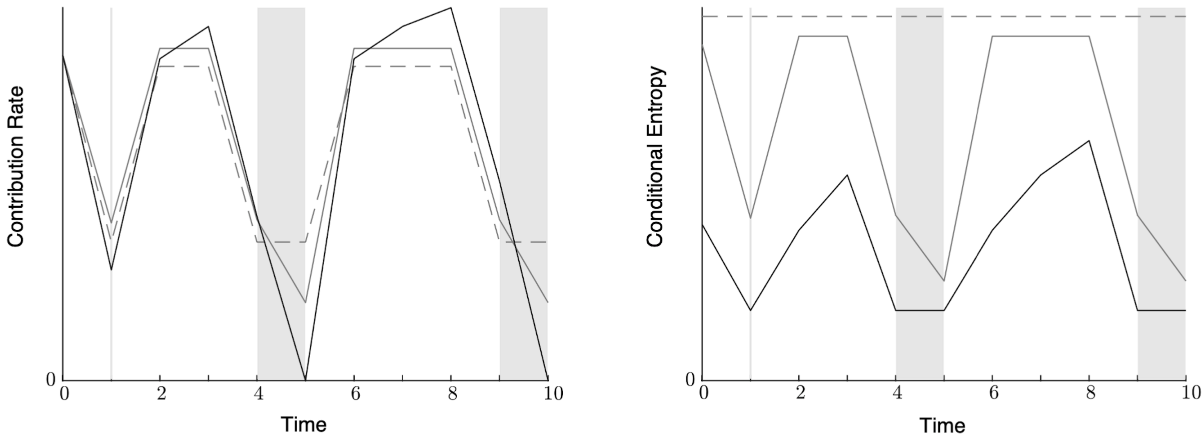


Figure 9: Panel A – Contribution Rates. Panel B – Conditional Entropy.

Note. — In both panels, shaded areas mark periods in the low endowment state for the young ($s = 2$). The solid black line is the Approximated Rule, the dashed gray line is the Flat-rate Rule, and the solid gray line is the Contributory-related Rule.

Table 1 and Figure 9 show that the Approximated Rule does well in matching the welfare and risk properties of the optimal solution. The contribution rate increases whenever there is a sequence of consecutive states 1s, while it adjusts downward and hits the minimum level $a(2)$ in finite time whenever there is a sequence of consecutive state 2s. The associated conditional entropy increases with the number of consecutive state 1s, showing that the young born after such a sequence bear the greatest risk. In contrast, the young born in state 2 face the lowest risk regardless of the history of endowment shocks.

The Flat-rate and Contributory-related Rules generate different paths of contribution rates and risk faced by each generation. Under the Flat-rate Rule, the contribution rate fluctuates with the endowment state and the conditional entropy is constant, so that all generations face the same risk. Under the Contributory-related Rule and with $b(1) = 0$, contributions are linked to the past only in state 2. The contribution rate and the conditional entropy remain constant whenever state 1 occurs, while both decrease with the number of consecutive state 2s. Thus, the young born after an infinitely long sequence

[40] It is intuitive that $b(1) = 0$ for the Contributory-related Rule since in this case the participation constraint always binds in state 1 for all contribution rates. Although we numerically find that $b(1) = 0$ is optimal for the canonical example, this may not be the case for all examples: lowering $b(1)$ toward zero improves risk sharing over time in state 1, but the long-run distribution shifts to put more weight on higher contribution rates, thereby reducing the risk-sharing benefits in state 2.

of state 2s bear the lowest risk. The contribution rate of the rich young is higher and the contribution rate of the poor young is generally lower under the Contributory-related Rule than under the Flat-rate Rule. This raises welfare and reduces risk relative to the Flat-rate Rule, as shown in Table 1.

As stated in Section III, when enforceability is an issue, the optimal sustainable social security scheme has a contributory-related element. This section shows that a linear approximation for the contributory-related component can be close to optimal.

VIII Discussion

Comparison with Infinitely-lived Agents. — It is worthwhile contrasting our results to those on risk sharing and limited commitment with infinitely-lived agents. The case of two infinitely-lived agents with endowment risk has been considered by Thomas and Worrall (1988) and Kocherlakota (1996). A common feature of that model with the overlapping generations model considered here is that only one agent is ever constrained at any point in time, namely, the agent making the transfer or facing a non-negativity constraint. To illustrate the contrast between the two models, suppose there are just two i.i.d. endowment states in which the first agent has the higher endowment in one state, while the situation is reversed in the other state. In the case with two infinitely-lived agents, there is convergence to the invariant distribution immediately after both states have occurred. If there is partial insurance at the invariant distribution, then there are two different ratios of marginal utilities associated with the two endowment states. Since consumption is determined by the endowment state, the conditional and unconditional autocorrelation of consumption between two adjacent dates is zero and there is no persistence in consumption. In the overlapping generations model, if there is partial insurance, then convergence occurs but not within finite time. With two endowment states, each generation faces only two potential ratios of marginal utilities, but these ratios differ from one generation to the next depending on the previous promise. Hence, as shown in Section VI, there is autocorrelation between the consumption of the young of adjacent generations even in the long run.

Our overlapping generations model is more closely related to models of risk sharing and limited commitment with a continuum of infinitely-lived agents (see, for example, Thomas and Worrall, 2007; Krueger and Perri, 2011; Broer, 2013). In those models, agents have high or low income (employed or unemployed). If there is partial insurance, then there is a finite set of possible transfers from the employed to the unemployed at the invariant distribution. There are three different ratios of marginal utilities, one each for constrained agents whether employed or unemployed and one for unconstrained and unemployed agents. Each ratio determines a value of consumption that depends only on the employment state for constrained agents but varies with the spell of unemployment for unconstrained and unemployed agents. To maintain a constant growth rate of marginal utilities, the consumption of an unconstrained and unemployed agent varies over time. This contrasts to the overlapping generations model considered here, where consumption changes when the same state reoccurs and the young are constrained. Another difference is that we derive the optimal sustainable intergenerational insurance for any given promise and establish strong convergence to the invariant distribution, whereas Krueger and Perri (2011) and Broer (2013) consider the solution only at an invariant distribution and Thomas and Worrall (2007) discuss convergence only in a special case.

Pareto Optimality. — Two concepts of Pareto optimality are widely used in stochastic overlapping generations models: *ex ante* optimality and *interim* optimality. In an ex ante Pareto optimum, it is impossible to increase the expected lifetime utility of one generation without reducing that of another generation. When the planner weighs all generations equally, the ex ante optimum corresponds to the first-best outcome of Section II, which is called an *equal-treatment Pareto optimum* by Labadie (2004). In the interim case, generations are distinguished by the state as well as the date of birth and at an interim Pareto optimum it is impossible to increase the lifetime utility of one generation without decreasing the lifetime utility of another generation conditional on the current state. This concept is often referred to as conditional Pareto optimality (see, for example, Aiyagari and Peled, 1991).

The optimal sustainable intergenerational insurance we have characterized is conditionally Pareto optimal. The ratio of marginal utilities varies across endowment states because the participation constraint of the young binds in some states. This means that there is only partial sharing of endowment risk, unlike in an equal-treatment Pareto optimum. The ratio of marginal utilities is constant across all endowment states only when the participation constraint of the young does not bind in any endowment state, that is, when the optimal allocation is first best. Nevertheless, even when the first best cannot be sustained, the optimal sustainable intergenerational insurance has a built-in equal-treatment property with all generations born in the same state $x = (\omega, s)$ receiving the same conditional allocation.

To understand the Pareto weight that the planner places on the lifetime utility of each generation, reconsider condition (12) and for simplicity ignore the non-negativity constraint on transfers and the upper bound on the promised utility. In determining the optimum, the Pareto weight on the utility of the old born at time t can be interpreted to be $\delta^t \beta(1 + \nu(\omega))$, where ω is the promise they received when they were young. The young they interact with at time $t+1$ have a Pareto weight of $\delta^{t+1}(1+\mu(\omega,s))$, which reflects the promise made to the old and the endowment state. By the updating property of equation (15), $\nu(\omega') = \mu(\omega,s)$. If $\mu(\omega,s) < \nu(\omega)$, then $\omega' < \omega$ and the young are given a lower weight than the old. If, on the other hand, $\mu(\omega,s) > \nu(\omega)$, then $\omega' > \omega$ and the young are given a greater weight than the old. It is straightforward to establish that the Pareto weight for each generation in state x at the invariant distribution is $\varphi(x)(1+\mu(x))/(1+\sum_y \varphi(y)\mu(y))$ where $\varphi(x)$ is the probability of state x.[41] This shows that the optimal sustainable intergenerational insurance is conditionally Pareto optimal with the Pareto weights determined endogenously by the participation constraints and the history of shocks. Furthermore, although agents born in the same endowment states may be treated differently, each generation with the same promise and endowment is treated equally.

Decentralization. — The optimal sustainable intergenerational insurances involves a degree of ex ante risk sharing between generations. In terms of our canonical example, generations born in endowment state 1 after a long sequence of preceding and consecutive state 1s transfer more than generations also born in endowment state 1 after a similar but shorter sequence. As pointed out by Demange and Laroque (1999), such an allocation cannot be supported as a decentralized equilibrium since there is no market in which the young can buy insurance before they are born.[42] Denote the intertemporal budget of the young in state x by $\mathbb{B}(x) := -\tau(x) + \sum_{x'} q(x,x')\tau(x')$, where $x' = (f(x), s')$, $\tau(x)$ is the optimal transfer and $q(x,x')$ is the associated state price for the delivery of one unit in state x' next period at the invariant distribution. In a decentralized system, the transfer $\tau(x)$ can be identified as the price of an infinitely-lived asset (money), in which case $\mathbb{B}(x) \le 0$ for each x (see, for example, Rangel and Zeckhauser, 2000). The first-best allocation derived in Section II does not satisfy this constraint and $\mathbb{B}(s) > 0$ for some state s. Likewise, with the optimal sustainable intergenerational insurance, it cannot be the case that $\mathbb{B}(x) \ge 0$ for each state x, otherwise, there would be a set of Pareto improving transfers. However, there will generally be states for which $\mathbb{B}(x) > 0$. In our canonical example, $\mathbb{B}(\omega,1) < 0$ and $\mathbb{B}(\omega,2) > 0$ because the transfer is high in state 1 and low in state 2.[43]

[41] Peled (1984) and Labadie (2004), among others, show that at a stationary Pareto optimal solution $\delta\alpha(x')u_c(c^y(x')) = \beta\sum_x \alpha(x)\pi(x,x')u_c(e(s') - c^y(x'))$, where $\alpha(x)$ is the implied Pareto weight. The implied Pareto weight in our case is derived from this equation using the first-order condition in (12) and (15) under the assumption that the upper bound and non-negativity constraints do not bind.

[42] It is well known that the market equilibrium with an infinitely-lived asset is interim efficient (see, for example, Peled, 1984)

[43] Alvarez and Jermann (2000) introduced the idea of "not too tight" borrowing constraints into a model of limited commitment with infinitely-lived agents. In this case, the two agents may have different marginal rates of substitutions at the optimum. The state price is set equal to the maximum but, in order

Robustness and Extensions. — We have kept the model as simple as possible for the sake of readability and tractability. In particular, we have assumed that the economic environment is stationary to emphasize that the dynamics of the model derive from the participation constraints themselves rather than from the underlying endowment process. Nevertheless, the results are extendable in several directions. We briefly discuss some of these extensions here and provide further details in Part G of the Supplementary Appendix.

It has been assumed that the old and the young have a common utility function, but this is inessential and easily generalized. Likewise, the analysis can be adapted to allow for stochastic growth of the endowments, along the lines of Alvarez and Jermann (2001) or Krueger and Lustig (2010), among others. In the basic model, there is no storage technology or savings. It is possible to allow for storage possibilities, provided that the rate of return to storage is not too high since, in that case, storage will not be used at the optimum. The model can also be adapted to the case where the old care altruistically about the young. For example, if the old attach a weight of $\varsigma > 0$ to the utility of the young, then the properties of Lemmas 2 and 3 continue to hold, provided that ς is not too large. It is also possible to allow for some persistence in the endowment process. For example, it can be shown that the shooting algorithm that was used in the two-state case of Section VI, can be generalized to the case where there is some persistence of the endowment process, provided that the persistence is not too large.

The optimal sustainable intergenerational insurance is not renegotiation-proof despite belonging to the Pareto frontier of the set of all equilibrium payoffs. This is because the young receive their autarkic utility when they default. In the case of default, it would be possible to offer the promised utility ω_0 instead of $\omega_{\min}$ without diminishing the planner's payoff. A simple modification of the participation constraint is needed to derive a renegotiation-proof outcome. Replace the participation constraint of the young in equation (8) by the inequality $u(c^y(s)) + \beta\omega'(s) \geq u(e^y(s)) + \beta\omega_0$. Since ω_0 is determined as part of the solution and appears in the constraint, a fixed point argument similar to that used by Thomas and Worrall (1994) is required to find the solution.[44] Although imposing this tighter constraint restricts risk sharing, the structure of the constrained optimization problem is not affected and we expect that the qualitative properties of the optimal solution are substantially unchanged.

to match the optimum, a (not too tight) constraint is imposed on the assets an agent can accumulate. In our overlapping generations model, the only relevant marginal rate of substitution is that of the young.

[44] In a deterministic overlapping generations model, Prescott and Ríos-Rull (2005) consider a similar condition, which they refer to as a no-restarting condition.

IX Conclusion and Future Research

The paper has developed a theory of intergenerational insurance in a stochastic overlapping generations model when risk-sharing transfers are voluntary. In this setting, it has been shown that (*i*) generational risk is spread across future generations in ways that create history dependence of transfers with periodic resetting, at which time the history of shocks is forgotten; (*ii*) there is heteroskedasticity of consumption conditional on the endowment and autocorrelation of consumption between adjacent generations, even in the long run; and (*iii*) the optimum can be interpreted as a pay-as-you-go social security scheme with both flat-rate and contributory-related elements along with means testing.

The results suggest several potential directions for future research. Firstly, the model is deliberately parsimonious in its assumptions to highlight the role played by the limited enforcement of transfers. Despite the stationarity of the underlying economic environment, limited enforcement generates heteroskedasticity and autocorrelation of consumption. An alternative approach to explain rich dynamics in consumption is to assume that the underlying endowment or earnings process is itself non-stationary (see, for example, De Nardi, Fella, and Paz-Pardo, 2020). An important avenue for future research is to combine these two approaches to better understand the role of both earning dynamics and limited enforcement in determining consumption allocations.

Secondly, a key feature of the optimal social security scheme with limited enforcement of transfers is the existence of a threshold for the past contribution, above which the pension is contributory-related. While it might be tempting to interpret this to mean that agents making a higher contribution when young receive a larger pension, the model has no heterogeneity within a generation and the correct interpretation applies only to the contributions of different generations of the young. Therefore, another important extension is to enrich the demographic structure of the model, either by having more than two overlapping generations or allowing for heterogeneity within the same generation, thereby making it possible to address the interdependence between intergenerational and intra-generational insurance.

Finally, there is no storage or production technology in the model for transforming endowments from one date to another. Introducing such a technology would make it possible to study the interplay between self-insurance and intergenerational insurance.

Appendix

Proofs of Propositions 1 and 3 and the proof of Lemma 1 can be found in Part B of the Supplementary Appendix.

Proof of Lemma 2. It is established in the proof of Lemma 1 that the participation constraints of the young and old cannot bind simultaneously in a given endowment state. For convenience, define:

$$h(\omega) := -\tfrac{\delta}{\beta}V_\omega(\omega); \quad g^c(\omega) := \tfrac{\beta}{\delta}(1+h(\omega)); \quad v(\omega,\mu;e) := u(y(\omega,\mu;e)) + \beta h^{-1}(\mu-\xi);$$

where $y(\omega,\mu;e)$ and $\vartheta(\omega,e,\hat{v})$ are defined implicitly by:

$$\frac{g^c(\omega)}{1+\mu} = \frac{u_c(y(\omega,\mu;e))}{u_c(e-y(\omega,\mu;e))}; \;\; v(\omega,\vartheta(\omega;e,\hat{v});e) = \hat{v}; \;\; \text{and} \;\; g(\omega,\mu;e) := \frac{u_c(y(\omega,\mu;e))}{u_c(e-y(\omega,\mu;e))}.$$

The term $y(\omega,\mu;e)$ is the consumption of the young given ω, the multiplier μ and the aggregate endowment e. Likewise, $v(\omega,\mu;e)$ is the lifetime utility of the young and $\vartheta(\omega,e,\hat{v})$ is the value of μ when the participation constraint of the young binds. Recall that $\omega_0 = \sup\{\omega \mid V_\omega(\omega) = 0\}$. The function $h\colon\Omega \to [0,\nu_{\max}]$ is strictly increasing for $\omega > \omega_0$ with $h(\omega_{\max}) = \nu_{\max}$ and $h(\omega) = 0$ for $\omega \leq \omega_0$. From equation (13), $\omega' = h^{-1}(\mu-\xi)$. The function y is continuous by the implicit function theorem because the derivative u_c and the function h are continuous. It can be checked that y is increasing in e (with $\partial y/\partial e < 1$) and μ, and decreasing in ω. Recall that $\xi > 0$ only if $\omega' = \omega_{\max}$ and hence, $\mu \geq \nu_{\max}$. For $\mu = 0$ and hence, $\xi = 0$, $h^{-1}(0) = \omega_0$, $y(\omega_0, 0; e) = c^{y*}(s)$ and $v(\omega_0, 0; e) = u(c^{y*}(s)) + \beta\omega_0$. It follows from the properties of $v(\omega,\mu;e)$ that ϑ is increasing in ω (weakly because the solution may be $\mu = 0$) and $\hat{v}$, and decreasing in e. The function g is the ratio of the marginal utilities of the young and the old. It is decreasing in μ and increasing in ω.

(i) Since the constraint set Φ is convex and the objective function is strictly concave, the policy function $f(\omega,s)$ is single-valued and continuous in ω. It is also non-decreasing in ω. It follows from the definitions above that $\omega' = f(\omega,s) = \min\{h^{-1}(\vartheta(\omega;e,\hat{v})),\omega_{\max}\}$. For $\vartheta = 0$, $f(\omega,s) = \omega_0$. For $\vartheta > 0$, $h^{-1}(\vartheta(\omega;e,\hat{v}))$ is strictly increasing in ω and hence, $f(\omega,s)$ is strictly increasing in ω provided $f(\omega,s) < \omega_{\max}$. If $\xi(\omega,s) > 0$ (or equivalently, $h^{-1}(\vartheta(\omega;e,\hat{v})) > \omega_{\max}$), then $f(\omega,s) = \omega_{\max}$.

(ii) The value of the threshold $\omega^c(e,\hat{v})$, above which μ is positive, is determined by $v(\omega^c(e,\hat{v}),0;e) = \hat{v}$. Thus, $\omega^c(e,\hat{v})$ is increasing in e and decreasing in $\hat{v}$. We next show that there is some state r such that $\omega^c(e(r),\hat{v}(r)) > \omega_0$. Taking $r = S$, it can be shown that $\mu(\omega_0, S) = 0$. To see this, suppose to the contrary that $\mu(\omega_0, S) > 0$. Then, $\eta(\omega_0, S) = 0$ and $g(\omega_0, \mu(\omega_0, S), e(S)) < \beta/\delta = g^c(\omega_0)$. Since there is no transfer from the old, $y(\omega_0, \mu(\omega_0, S), e(S)) \leq e^y(S)$ and hence, $g(\omega_0, \mu(\omega_0, S), e(S)) \geq \lambda(S)$. By Assumption 3, $\lambda(S) \geq \beta/\delta$, which gives a contradiction. Since $\omega_0 > \omega_{\min}$, $v(\omega_0, 0; e(S)) = u(e^y(S)) + \beta\omega_0 > u(e^y(S)) + \beta\omega_{\min} = \hat{v}(S)$. Finally, since $v(\omega, 0; e)$ is continuous and decreasing in ω and $v(\omega^c(e(S), \hat{v}(S)), 0; e(S)) = \hat{v}(S)$, it follows that $\omega^c(e(S), \hat{v}(S)) > \omega_0$, as required.

(iii) Existence of a fixed point $\omega^f(s)$ of the mapping $f(\omega, s)$ follows from the standard fixed point theorem given the continuity and monotonicity of $f(\omega, s)$ in ω. Part (*ii*) shows that there is at least one state, namely $s = S$, for which $\mu(\omega_0, s) = 0$. By Lemma 1, $\omega_0 < \omega^*$ and hence, at least one of the participation constraints of the young binds at $\omega = \omega_0$. Thus, there is at least one state $r \in \mathcal{S}$ such that $\mu(\omega_0, r) > 0$. It follows that the set of states can be partitioned into two non-empty subsets, $\hat{\mathcal{S}}$ and its complement with $\mu(\omega_0, s) = 0$ for $s \in \hat{\mathcal{S}}$ and $\mu(\omega_0, r) > 0$ for $r \notin \hat{\mathcal{S}}$.

For states $s \in \hat{\mathcal{S}}$, $\mu(\omega_0, s) = \xi(\omega_0, s) = 0$ and hence, $f(\omega_0, s) = h^{-1}(0) = \omega_0$. That is, ω_0 is a fixed point of the mapping $f(\omega, s)$. Since $f(\omega, s) \geq \omega_0$ (see, part (*i*)), there can be no $\omega^f(s) < \omega_0$. Now consider a state $s \notin \hat{\mathcal{S}}$ where $\mu(\omega_0, s) > 0$. Since $\nu(\omega_0) = 0$ and $\eta(\omega_0, s) = 0$ by complementary slackness, it follows that $f(\omega_0, s) > \omega_0$ (with $f(\omega_0, s) = \omega_{\max} > \omega_0$ if $\xi(\omega_0, s) > 0$). That is, in any state where $\mu(\omega_0, s) > 0$, any fixed point satisfies $\omega^f(s) > \omega_0$. First, note that for $\omega^f(s) > \omega_0$, $\mu(\omega^f(s), s) = \nu(\omega^f(s)) + \xi(\omega^f(s), s)$. If $\xi(\omega^f(s), s) > 0$, then $\nu(\omega^f(s)) = \nu_{\max}$ and $\omega^f(s) = \omega_{\max}$. Then, from condition (12), $c^y(\omega_{\max}, s) < c^{y*}(s)$ and $u(c^y(\omega_{\max}, s)) = u(e^y(s)) - \beta(\omega_{\max} - \omega_{\min})$. If $\xi(\omega^f(s), s) = 0$, then $\mu(\omega^f(s), s) = \nu(\omega^f(s))$ and hence, $c^y(\omega^f(s), s) = c^{y*}(s)$. Hence, $\omega^f(s) = \omega_{\min} + \beta^{-1}(u(e^y(s)) - u(c^{y*}(s)))$. Taking the cases $s \in \hat{\mathcal{S}}$ and $s \notin \hat{\mathcal{S}}$ together, we obtain:

$$\omega^f(s) = \min\left\{\max\left\{\omega_0, \omega_{\min} + \beta^{-1}\left(u(e^y(s)) - u(c^{y*}(s))\right)\right\}, \omega_{\max}\right\}. \tag{21}$$

From Proposition 2, $c^{y*}(s)$ is unique and hence, from equation (21) it follows that $\omega^f(s)$ is unique. There may, of course, be multiple states with the same fixed point.

(iv) Recall that $\omega'(\omega, e, \hat{v}) = \min\{h^{-1}(\vartheta(\omega, e, \hat{v})), \omega_{\max}\}$. It follows from the properties of h and ϑ derived above that $\omega'(\omega, e, \hat{v})$ is decreasing in e and increasing in $\hat{v}$. To determine how $f(\omega, s)$ depends on s, we need to know how $e(s)$ and $\hat{v}(s)$ depend on s. When e is fixed, it follows from the convention on $\lambda(s)$ that $e^y(1) \geq e^y(2) \geq \cdots \geq e^y(S)$. Hence, $\hat{v}(s)$ is decreasing in s. Moreover, for distinct states s and r with $s < r$ and $e^y(s) > e^y(r)$, we have $\hat{v}(s) > \hat{v}(r)$. Thus, $f(\omega, s) = \min\{h^{-1}(\vartheta(\omega, e, \hat{v}(s))), \omega_{\max}\}$ is decreasing in s. Moreover, for distinct s and r such that both $f(\omega, s) \in (\omega_0, \omega_{\max})$ and

$f(\omega, r) \in (\omega_0, \omega_{\max})$, $f(\omega, s) > f(\omega, r)$ for $e^y(s) > e^y(r)$. We can order fixed points: $\omega^f(s) \geq \omega^f(r)$ for every $s < r$, with strict inequality unless $\omega^f(s) = \omega^f(r) = \omega_0$ or $\omega^f(s) = \omega^f(r) = \omega_{\max}$. QED

Proof of Lemma 3.

(i) It is established in part (i) of Lemma 2 that the function $y(\omega, \mu; e)$ is strictly decreasing in ω for a fixed μ and e, provided that the non-negativity constraint on transfers does not bind. If $\mu(\omega, s) > 0$, then $u(c^y(\omega, s)) + \beta f(\omega, s) = \hat{v}(s)$. Since Lemma 2 establishes that $f(\omega, s)$ is strictly increasing in ω for $f(\omega, s) \in (\omega_0, \omega_{\max})$, it follows that $c^y(\omega, s)$ is strictly decreasing in ω for $c^y(\omega, s) < e^y(s)$ and $f(\omega, s) < \omega_{\max}$.

(ii) This is shown in the proof of part (iii) of Lemma 2.

(iii) For a fixed aggregate endowment e, the lifetime endowment utility $\hat{v}(s)$ is decreasing in s. If the participation constraint of the young binds, then from Lemma 2, $\vartheta(\omega; e, \hat{v})$ is increasing in $\hat{v}$. Since $y(\omega, \vartheta(\omega; e, \hat{v}); e)$ is increasing in ϑ, $c^y(\omega, s)$ is increasing in $\hat{v}$ and hence, decreasing in s. It is constant in s if the participation constraint of neither the young nor the old binds. QED

Proof of Corollary 1.

(i) Since $g^z(\omega, s) = u(e^y(s)) - u(c^y(\omega, s))$, it follows from Lemma 3 that $g^z(\omega, s)$ is continuous and increasing in ω and strictly increasing for $g^z(\omega, s) > 0$. Lemma 2 implies that $f^z(z^-) = \omega_0$ if $z^- \leq z^c$ and $f^z(z^-) = \omega_{\min} + z^-/\beta$ otherwise. Hence, from equation (16), $h^z(z^-, s)$ is continuous and increasing in z^- and strictly increasing for $h^z(z^-, s) > 0$.

(ii) Consider two states s and $\hat{s}$ with $\hat{s} > s$. It follows from Lemma 2 that with a fixed aggregate endowment e, $g^z(\cdot, s) \geq g^z(\cdot, \hat{s})$. Then, from equation (16), $u(e^y(s)) - u(e^y(s) - h^z(\cdot, s)) \geq u(e^y(\hat{s})) - u(e^y(\hat{s}) - h^z(\cdot, \hat{s}))$ where $h^z(\cdot, s) \geq 0$ and $h^z(\cdot, \hat{s}) \geq 0$. Since $e^y(s) > e^y(\hat{s})$ for a fixed e and since u is strictly concave, $h^z(\cdot, s) \geq h^z(\cdot, \hat{s})$, with strict inequality if $h^z(\cdot, s) > 0$. QED

Proof of Proposition 4. Using Lemma 2 and the argument in the text, it can be seen that there is an $k \geq 1$ and $\epsilon > 0$ such that $P^k(\omega, \{\omega_0\}) > \epsilon$ for all $\omega \in [\omega_0, \bar{\omega}]$ and it follows that Condition **M** of Stokey and Lucas (1989, page 348) is satisfied. Therefore, Theorem 11.12 of Stokey and Lucas (1989) applies and there is strong convergence. Non-degeneracy follows from Assumption 4 and existence of a mass point at ω_0 follows from Part (ii) of Lemma 2. The relationship between the probability mass and the expected return times is standard as is pointwise convergence (see, for example, Theorems 10.2.3 and 13.1.2 of Meyn and Tweedie, 2009). QED

Proof of Proposition 5.

(i) To simplify notation, let $m(\omega, s, s') := m((\omega, s), (f(\omega, s), s'))$. It follows from equation (17) and the monotonicity of $f(\omega, s)$ in ω that $m(\omega, s, s')$ is decreasing in ω. The price of a one-period discount bond in state (ω, s) is $p^1(\omega, s) = \sum_{s'} \pi(s')m(\omega, s, s')$, which is decreasing in ω. Making the induction hypothesis that the price of a k-period discount bond is decreasing in ω, $p^{k+1}(\omega, s) = \sum_{s'} \pi(s')m(\omega, s, s')p^k(f(\omega, s), s')$. Since $p^k(\omega, s)$ and $m(\omega, s, s')$ are positive and decreasing in ω, and $f(\omega, s)$ is increasing in ω, it follows that $p^{k+1}(\omega, s)$ is decreasing in ω. Hence, the conditional yield $y^k(\omega, s) = -\log(p^k(\omega, s))$ and the deviation $\Delta y^k(\omega, s)$ are increasing in ω.

(ii) This is a standard result (see, for example, Martin and Ross, 2019).

(iii) It follows from part (*ii*) that $\lim_{k\to\infty} y^k(x) = \mathbb{E}_\varphi[\log(m(x, x'))] = \log(\rho)$, where $\mathbb{E}_\varphi$ is the expectation taken over the invariant distribution of x and ρ is the Perron root of the matrix Q. Taking logs of equation (17) gives:

$$\log\left(m(x, x')\right) = \log\left(\beta\right) - \log\left(u_c(c^y(x))\right) + \log\left(u_c(c^o(x'))\right).$$

Furthermore, taking logs in condition (12) and moving it one period forward gives:

$$\log\left(u_c(c^y(x'))\right) - \log\left(u_c(c^o(x'))\right) = \log\left(\frac{\beta}{\delta}\right) + \log\left(1 + \nu(\omega') + \eta(x')\right) - \log\left(1 + \mu(x')\right).$$

Combining these two equations and using the updating property $\nu(\omega') = \mu(x) - \xi(x)$ gives:

$$\begin{aligned}\log\left(m(x, x')\right) = \log\left(\delta\right) &+ \log\left(u_c(c^y(x'))\right) - \log\left(u_c(c^y(x))\right) + \log\left(1+\mu(x')\right) - \log\left(1+\mu(x)\right)\\ &- \left(\log\left(1+\mu(x)-\xi(x)+\eta(x')\right) - \log\left(1+\mu(x)\right)\right).\end{aligned}$$

Assume that $\xi(x) = \eta(x') = 0$ for all x. Then, the last term in the above equation is zero and $\mathbb{E}_\varphi[\log(m(x, x'))] = \log(\delta)$. Hence, comparing to the standard result given in part (*ii*), when $\xi(x) = \eta(x') = 0$ for all x, $\rho = \delta$. If $\xi(x) = 0$ for each x, then $\rho \leq \delta$. If $\eta(x) = 0$ for each x, then $\rho \geq \delta$, with strict inequality if $\xi(x) > 0$ for some x.

(iv) With no aggregate risk, $\omega^f(s)$ is ordered by state. In particular, $\bar{\omega} = \omega^f(1) > \omega^f(S) = \omega_0$. Consider state (ω_0, S). Since ω_0 is a fixed point of the mapping $f(\omega, S)$ and $\mu(\omega_0, S) = 0$, it follows from equations (12) and (17) that $m(\omega_0, S, S) \geq \delta$, with equality if $\eta(\omega_0, S) = 0$. Furthermore, it follows from equation (17) that $m(\omega, s, s')$ is decreasing in s'. Hence, for each $s < S$, $m(\omega_0, S, s) \geq m(\omega_0, S, S) \geq \delta$, with at least one of the inequalities strict. Taking expectations, the bond price $p^1(\omega_0, S) > \delta$. Consequently, the yield $y^1(\omega_0, S) < -\log(\delta)$. Similarly, it can be checked that $m(\bar{\omega}, 1, 1) \leq \delta$, with equality if $\xi(\bar{\omega}, 1) = 0$, that is, if $\bar{\omega} < \omega_{\max}$. Hence, $m(\bar{\omega}, 1, s) \leq \delta$ for each $s > 1$, with strict inequality for some state, and consequently, $y^1(\bar{\omega}, 1) > -\log(\delta)$. QED

Proof of Proposition 6.

(i) It follows from parts (i) and (iii) of Lemma 3 that the functions $c^y(\omega, s)$ are monotonically decreasing in ω and ordered by the state: $c^y(\omega, 1) > c^y(\omega, 2)$. Since $\bar{\omega}$ and ω_0 are fixed points of $f(\omega, 1)$ and $f(\omega, 2)$, it follows from part (ii) of Lemma 3 that $c^y(\bar{\omega}, 1) = c^{y*}(1)$ and $c^y(\omega_0, 2) = c^{y*}(2)$. With no aggregate risk, the first-best consumption is independent of the state, $c^{y*}(1) = c^{y*}(2)$, and hence, $c^y(\bar{\omega}, 1) = c^y(\omega_0, 2)$.

(ii) It follows from condition (12), the definition of the stochastic discount factor in equation (17) and the Ross Recovery Theorem that $\rho^B = 1$ and $\psi^B(x) = 1/(1 + \mu(x))$. Hence, from Proposition 5, the Perron root of the state price matrix Q is $\rho = \rho^A = \delta$. Let $\upsilon(\omega) := 1 + \mu(\omega, 1)$. Since $\mu(\omega, 2) = 0$, it follows that:

$$\psi^A(\omega, 1) = \frac{1}{u_c(c^y(\omega, 1))}; \;\; \psi^B(\omega, 1) = \upsilon(\omega)^{-1}; \;\; \psi^A(\omega, 2) = \frac{1}{u_c(c^y(\omega, 2))}; \;\; \psi^B(\omega, 2) = 1.$$

Using CRRA utility gives the specification in the statement of the proposition and the consumption of the young is determined by:

$$c^y(f(\omega, 1), 1) = \frac{\upsilon(f(\omega, 1))^{\frac{1}{\gamma}}}{\upsilon(f(\omega, 1))^{\frac{1}{\gamma}} + \left(\frac{\beta}{\delta}\upsilon(\omega)\right)^{\frac{1}{\gamma}}}; \quad c^y(f(\omega, 1), 2) = \frac{1}{1 + \left(\frac{\beta}{\delta}\upsilon(\omega)\right)^{\frac{1}{\gamma}}};$$

$$c^y(\omega_0, 1) = \frac{\upsilon(\omega_0)^{\frac{1}{\gamma}}}{\upsilon(\omega_0)^{\frac{1}{\gamma}} + \left(\frac{\beta}{\delta}\right)^{\frac{1}{\gamma}}}; \;\; c^y(\omega_0, 2) = c^y(\bar{\omega}, 1) = c^{y*}(2) = c^{y*}(1) = \frac{1}{1 + \left(\frac{\beta}{\delta}\right)^{\frac{1}{\gamma}}}.$$

Since $f(\omega, 1)$ is increasing in ω, it follows from (13) that $\upsilon(\omega)$ is also monotonically increasing in ω and hence, $\psi^B(\omega, 1)$ is decreasing in ω. Likewise, it follows from parts (i) that both $\psi^A(\omega, 1)$ and $\psi^A(\omega, 2)$ are decreasing in ω and from the equation above that $\psi^A(\bar{\omega}, 1) = \psi^A(\omega_0, 2)$. Also, $\psi(\omega, 1) < \psi(\omega, 2)$ with $\psi(\omega, 1)$ and $\psi(\omega, 2)$ decreasing in ω. Hence, $\psi_{\max} = \psi(\omega_0, 2) = 1/u_c(c^{y*}(2))$ and $\psi_{\min} = \psi(\bar{\omega}, 1) = 1/(u_c(c^{y*}(1))\upsilon(\bar{\omega}))$. With no aggregate risk, $c^{y*}(1) = c^{y*}(2)$ and hence, $\Upsilon = \log(\psi_{\max}/\psi_{\min}) = \log(\upsilon(\bar{\omega}))$. Likewise, $\psi^B_{\max} = 1$ and $\psi^B_{\min} = \upsilon(\bar{\omega})^{-1}$, so that $\Upsilon^B = \log(\psi^B_{\max}/\psi^B_{\min}) = \log(\upsilon(\bar{\omega})) = \Upsilon$. To compute $\upsilon(\bar{\omega})$, we use the fact that both the participation constraint of the young in state 1 and the promise-keeping constraint bind at $\omega = \bar{\omega}$, that is,

$$\beta\left(\pi\left(u\left(c^o(\bar{\omega},1)\right)-u\left(e^o(1)\right)\right)+(1-\pi)\left(u\left(c^o(\bar{\omega},2)\right)-u\left(e^o(2)\right)\right)\right)=u\left(e^y(1)\right)-u\left(c^y(\bar{\omega},1)\right).$$

From this, with CRRA utility and the consumption values given above, we get:

$$\Upsilon = \log\left(v(\bar{\omega})\right) = \log(\delta) - \log(\beta) - \gamma\log\left(\Xi_\gamma^{\frac{1}{\gamma-1}} - 1\right), \quad \text{where}$$

$$\begin{aligned}\Xi_\gamma = {} & \left(\tfrac{1}{\beta(1-\pi)}\right)(\kappa+\sigma)^{1-\gamma} + \beta\left(\tfrac{\pi}{1-\pi}\right)(1-\kappa-\sigma)^{1-\gamma} + \left(1-\kappa+\sigma\tfrac{\pi}{1-\pi}\right)^{1-\gamma} \\ & - \left(\tfrac{1}{\beta(1-\pi)}\right)\left(\left(\frac{1}{1+\left(\frac{\beta}{\delta}\right)^{\frac{1}{\gamma}}}\right)^{1-\gamma} + \beta\pi\left(\frac{\left(\frac{\beta}{\delta}\right)^{\frac{1}{\gamma}}}{1+\left(\frac{\beta}{\delta}\right)^{\frac{1}{\gamma}}}\right)^{1-\gamma}\right).\end{aligned}$$

In the limit as $\gamma \to 1$, we have:

$$\Xi_1 = \left(\frac{\delta}{\beta}\right)^{\frac{\pi}{1-\pi}}\left(\frac{\beta+\delta}{\delta}\right)^{\frac{1+\beta\pi}{\beta(1-\pi)}}(\kappa+\sigma)^{\frac{1}{\beta(1-\pi)}}(1-\kappa-\sigma)^{\frac{\pi}{1-\pi}}\left(1-\kappa+\sigma\frac{\pi}{1-\pi}\right).$$

(iii) In the two-state case, conditional entropy can be defined as:

$$L^i(\omega,s)=\log\left(\pi m^i(\omega,s,1)+(1-\pi)m^i(\omega,s,2)\right)-\pi\log(m^i(\omega,s,1))-(1-\pi)\log(m^i(\omega,s,2)),$$

for $i = A, B$. From the Ross Recovery Theorem, $m^i(\omega, s, s') = \rho^i\psi^i(\omega, s)/\psi^i(f(\omega, s), s')$. In state 2, $f(\omega, 2) = \omega_0$. Therefore, using the eigenvectors defined above gives the formulas in the statement of the proposition. QED

Proof of Proposition 7.

(i) Since the promised utility ω is reset to ω_0 whenever state 2 occurs, the probability that the promised utility is ω_0 is $1-\pi$, irrespective of the date or history. Therefore, T periods after such a resetting, the distribution of ω is:

$$\phi_T\left(\omega_0^{(n)}\right) = (1-\pi)\pi^n \quad \text{for} \quad n = 0,1,2,3,\ldots,T-1 \quad \text{and} \quad \phi_T\left(\omega_0^{(T)}\right) = \pi^T.$$

The distribution ϕ_T satisfies the recursion

$$\phi_{T+1}(\omega_0) = (1-\pi)\sum_{n=0}^{T} \phi_T\left(\omega_0^{(n)}\right) \quad \text{and}$$
$$\phi_{T+1}\left(\omega_0^{(n+1)}\right) = \pi\phi_T\left(\omega_0^{(n)}\right) \quad \text{for} \quad n = 0,1,2,3,\ldots,T.$$

In the limit, ϕ_T converges to the invariant distribution $\phi(\omega_0^{(n)}) = (1-\pi)\pi^n$ for $n = 0,1,\ldots,\infty$, which is a simple geometric distribution. The expected time to regeneration is $\varpi = (1-\pi)\sum_{j=1}^{\infty} j\pi^{j-1} = (1-\pi)(1+2\pi+3\pi^2+\cdots) = 1/(1-\pi)$. Denote the expected return times for $\omega_0^{(n)}$ by ϖ_n. A first-step analysis of the Markov chain shows that $\varpi_n = 1/(\pi^n(1-\pi)) = \varpi/\pi^n$. The invariant distribution $\varphi(x)$ is easily calculated from ϕ because $\varphi(\omega,s) = \pi(s)\phi(\omega)$.

(ii) Mean entropy is computed from the conditional entropy given in Proposition 6 and the invariant distribution derived in part (i).

(iii) From the first-order condition (12), $\log(m^B(\omega,2)) = \log(m^B(\omega,1)) - \log(\upsilon(\omega))$. The two variables $\log(m^B(\omega,1))$ and $-\log(\upsilon(\omega))$ are co-monotonic increasing in ω. Therefore, it follows by applying Chebyshev's order inequality that their covariance is positive. Computing the variance at the invariant distribution gives:

$$\mathrm{var}(\log(m^B(\omega,2))) = \mathrm{var}(\log(m^B(\omega,1))) + \mathrm{var}(\log(\upsilon(\omega))) + \mathrm{cov}(\log(m^B(\omega,1)), -\log(\upsilon(\omega)))$$

and hence, $\mathrm{var}(\log(m^B(\omega,2))) > \mathrm{var}(\log(m^B(\omega,1)))$.

(iv) Using the invariant distribution, the conditional expected promised utility for $t+1$ is $\mathbb{E}[\omega_{t+1} \mid \omega_t = \omega_0^{(n)}] = (1-\pi)\omega_0 + \pi\omega_0^{(n+1)}$. Since $\omega_0^{(n)}$ is monotonically increasing in n, so too is the above conditional expectation. Thus, ω_t and $\mathbb{E}[\omega_{t+1} \mid \omega_t])$ are co-monotonic. Since $\mathrm{cov}(\omega_t,\omega_{t+1}) = \mathrm{cov}(\omega_t, \mathbb{E}[\omega_{t+1} \mid \omega_t])$, it follows that $\mathrm{cov}(\omega_t,\omega_{t+1}) > 0$.

(v) The argument of part (iv) can be applied to the conditional auto-covariance. Consumption $c^y(\omega,s)$ is decreasing in ω from Proposition 6. The expectation of the consumption of the young next period conditional on the current endowment state is:

$$\mathbb{E}[c_{t+1}^y \mid c_t^y = c^y(\omega,1)] = \pi c^y(f(\omega,1),1) + (1-\pi)c^y(f(\omega,1),2),$$
$$\mathbb{E}[c_{t+1}^y \mid c_t^y = c^y(\omega,2)] = \pi c^y(\omega_0,1) + (1-\pi)c^y(\omega_0,2).$$

Since the first expectation is decreasing in ω, $\mathrm{cov}(c_t^y, c_{t+1}^y \mid s_t = 1) > 0$. The second expectation is independent of ω and hence, $\mathrm{cov}(c_t^y, c_{t+1}^y \mid s_t = 2) = 0$. QED

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Supplementary Appendix — For Online Publication Only

This appendix presents supplementary material referenced in the paper. Part A provides evidence on the relative income of the young and the old for six OECD countries referred to in footnote 2 in the Introduction. Part B provides proofs of Propositions 1 and 3 from Sections I and II together with the proof of Lemma 1 from Section III. Part C derives results on the measures of generational risk stated in Section V. Part D presents the shooting algorithm used to derive the optimal allocation in Section VI. Part E provides comparative static results for the two-state example of Section VI. Part F examines two alternative welfare measures, the insurance coefficient and consumption-equivalent welfare change measure. Part G provides further details of the robustness and possible extensions of the model discussed in Section VIII. Part H describes the pseudo-code for the numerical algorithms used in the paper.

A Change in Relative Income of Young and Old

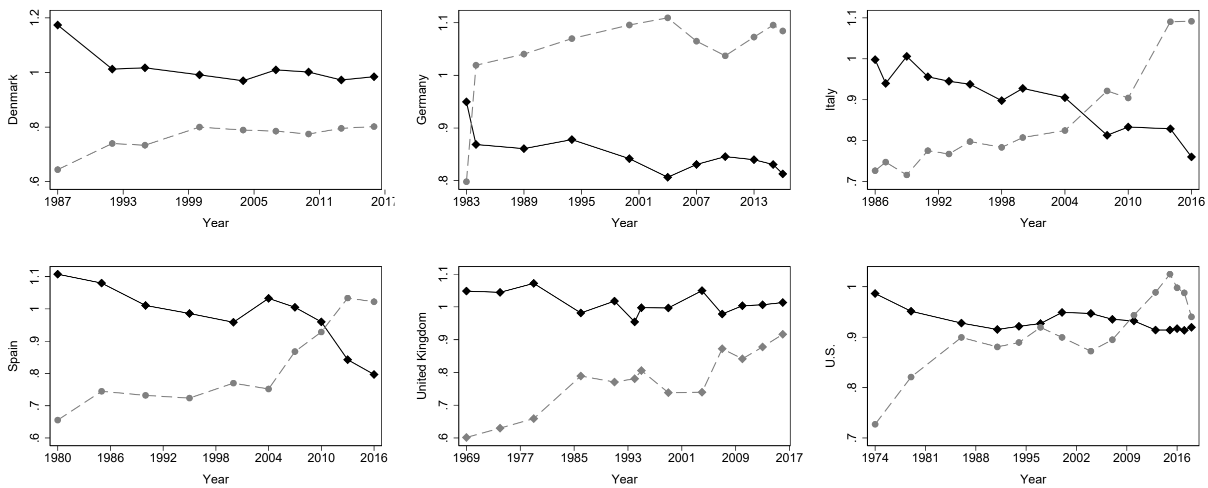

Figure A.1: Relative Income of Young and Old for six OECD Countries

Note. — The solid line is the average net (of taxes and transfers) equivalized disposable income for individuals aged 25-34 divided by the average of the same measure for the whole population. The dotted line is the corresponding ratio for individuals aged 65-74.

Figure A.1 illustrates the average disposable income of individuals aged 25-34 (the young) and the average disposable income of individuals aged 65-74 (the old) relative to the national average over recent decades for Denmark, Germany, Italy, Spain, U.K. and U.S. (data periods are country specific). Data is taken from the Luxembourg Income Study Database available at `www.lisdatacenter.org`. In each country there has been an improvement in the average disposable income of the old compared to the average disposable income of the young over the sample period. For example, the average disposable income of the young in the U.S. has fallen from just below the national average to just above 90% of the national average during 1974-2018. Over the same period, the old have fared much better with their average disposal income rising from approximately 70% of the national average to become roughly equal to the national average. Moreover, the old overtook the young for the first time around the time of the financial crisis of 2008.

A similar pattern can be seen in Italy and Spain and a narrowing of the gap between the young and the old can also be observed in Denmark and the U.K. Germany is somewhat different with the old overtaking the young as early as the 1980s.

B Proof of Propositions 1 and 3 and Lemma 1

Proof of Proposition 1. The lifetime endowment utility of an agent born in state r is:

$$\hat{v}(r) := u(e^y(r)) + \beta \sum_s \pi(s)u(e^o(s)).$$

Consider a small transfer $\mathrm{d}\tau(r)$ in state r from the young to the old. The problem of existence of a sustainable allocation can be answered by finding a vector of positive transfer $\mathrm{d}\tau$ such that there is a weak improvement over the lifetime endowment utility in all states and a strict improvement in at least one state. The change in the lifetime endowment utility induced by a vector $\mathrm{d}\tau$ is non-negative if

$$-u_c(e^y(r))\mathrm{d}\tau(r) + \beta \sum_s \pi(s)u_c(e^o(s))\mathrm{d}\tau(s) \geq 0. \tag{B.1}$$

Rearranging (B.1) in terms of the marginal rates of substitution $\hat{m}(r,s)$, we have:

$$-\mathrm{d}\tau(r) + \sum_s \pi(s)\hat{m}(r,s)\mathrm{d}\tau(s) \geq 0.$$

The problem of existence can then be addressed by finding a vector $\mathrm{d}\tau > 0$ that solves:

$$\left(\hat{Q} - I\right)\mathrm{d}\tau \geq 0, \tag{B.2}$$

where I is the identity matrix and $\hat{Q}$ is the matrix of $\hat{q}(r,s) = \pi(s)\hat{m}(r,s)$. Equation (B.2) has a well-known solution. Using the Perron-Frobenius theorem, there exists a strictly positive solution for $\mathrm{d}\tau$, provided that the Perron root, that is, the largest eigenvalue of $\hat{Q}$, is greater than one. This is satisfied by Assumption 2, which guarantees the existence of positive transfers from the young to the old that improve the utility of each generation. QED

Proof of Proposition 3. Define the critical transfer τ_1^c by:

$$u(e^y - \tau_1^c) + \beta u(e^o + \tau^*) = \hat{v} := u(e^y) + \beta u(e^o).$$

Define τ_n^c recursively by:

$$u(e^y - \tau_n^c) + \beta u(e^o + \tau_{n-1}^c) = \hat{v} \quad \text{for } n = 2, 3, \ldots, \infty.$$

From the strict concavity of utility function u, $\tau_n^c > \tau_{n-1}^c$ and $\lim_{n\to\infty} \tau_n^c = \bar{\tau} = e^y - e^o$. Correspondingly, define $\omega_n^c := u(e^o + \tau_n^c)$. We have $\omega_0^c = \omega^*$ and $\lim_{n\to\infty} \omega_n^c = \bar{\omega}$. Let $v^* = u(e^y - \tau^*) + (\beta/\delta)\omega^*$. With some abuse of notation, let $V_n(\omega)$ denote the value function when $\omega \in (\omega_{n-1}^c, \omega_n^c]$. Hence,

$$V_n(\omega) = u(e - u^{-1}(\omega)) + \frac{\beta}{\delta}\omega + \delta V_{n-1}\left(\frac{1}{\beta}\left(\hat{v} - u(e - u^{-1}(\omega))\right)\right).$$

For $\omega \leq \omega^*$, $\tau(\omega) = \tau^*$ and $\omega' = \omega^*$. Therefore, $V(\omega) = v^*/(1-\delta)$ for $\omega \in [u(e^o), \omega^*]$. For $\omega \in (\omega^*, \omega_1^c]$,

$$V_1(\omega) = u(e - u^{-1}(\omega)) + \frac{\beta}{\delta}\omega + \frac{\delta}{1-\delta}v^*.$$

Differentiating the function $V_1(\omega)$ gives:

$$\frac{\mathrm{d}V_1(\omega)}{\mathrm{d}\omega} = \frac{\beta}{\delta} - \frac{u_c(e - u^{-1}(\omega))}{u_c(u^{-1}(\omega))}.$$

Let $g(\omega) := u_c(e - u^{-1}(\omega))/u_c(u^{-1}(\omega))$. Since $\omega > \omega^*$, $g(\omega) > \beta/\delta$ and $\mathrm{d}V_1(\omega)/\mathrm{d}\omega < 0$. Note that $g(\omega^*) = \beta/\delta$ and therefore, in the limit as $\omega \to \omega^*$, $\mathrm{d}V_1(\omega)/\mathrm{d}\omega = 0$. Furthermore, the function $V_1(\omega)$ is strictly concave because $g(\omega)$ is increasing given the strict concavity of u. Using this result, we can proceed by induction and assume $V_{n-1}(\omega)$ is decreasing and strictly concave. Then, it is straightforward to establish that $V_n(\omega)$ is decreasing and strictly concave. Continuity follows since $\lim_{\omega\to\omega_n^c} V_{n+1}(\omega) = V_n(\omega_n^c)$. To establish differentiability, we need to demonstrate that:

$$\lim_{\omega\to\omega_n^c} \frac{\mathrm{d}V_{n+1}(\omega)}{\mathrm{d}\omega} = \frac{\mathrm{d}V_n(\omega_n^c)}{\mathrm{d}\omega}.$$

To show this, note that for $\omega \in (\omega_n^c, \omega_{n+1}^c)$:

$$\frac{\mathrm{d}V_{n+1}(\omega)}{\mathrm{d}\omega} = \frac{\beta}{\delta} - g(\omega)\left(1 - \frac{\delta}{\beta}\frac{\mathrm{d}V_n(\omega')}{\mathrm{d}\omega}\right).$$

Starting with $n = 1$, we have:

$$\lim_{\omega \to \omega_1^c} \frac{\mathrm{d}V_2(\omega)}{\mathrm{d}\omega} = \frac{\beta}{\delta} - g(\omega_1^c)\left(1 - \frac{\delta}{\beta}\lim_{\omega \to \omega_0^c} \frac{\mathrm{d}V_1(\omega)}{\mathrm{d}\omega}\right).$$

Since $\lim_{\omega \to \omega_0^c} \mathrm{d}V_1(\omega)/\mathrm{d}\omega = 0$, we have:

$$\lim_{\omega \to \omega_1^c} \frac{\mathrm{d}V_2(\omega)}{\mathrm{d}\omega} = \frac{\beta}{\delta} - g(\omega_1^c) = \frac{\mathrm{d}V_1(\omega_1^c)}{\mathrm{d}\omega}.$$

Therefore, make the recursive assumption that $\lim_{\omega \to \omega_{n-1}^c} \mathrm{d}V_n(\omega)/\mathrm{d}\omega = \mathrm{d}V_{n-1}(\omega_{n-1}^c)/\mathrm{d}\omega$. In general, we have:

$$\begin{aligned}\lim_{\omega \to \omega_n^c} \frac{\mathrm{d}V_{n+1}(\omega)}{\mathrm{d}\omega} &= \frac{\beta}{\delta} - g(\omega_n^c)\left(1 - \frac{\delta}{\beta}\lim_{\omega \to \omega_{n-1}^c} \frac{\mathrm{d}V_n(\omega)}{\mathrm{d}\omega}\right)\\ \frac{\mathrm{d}V_n(\omega_n^c)}{\mathrm{d}\omega} &= \frac{\beta}{\delta} - g(\omega_n^c)\left(1 - \frac{\delta}{\beta}\frac{\mathrm{d}V_{n-1}(\omega_{n-1}^c)}{\mathrm{d}\omega}\right).\end{aligned}$$

By the recursive assumption, these two equations are equal. Hence, we conclude that $V(\omega)$ is differentiable. In particular, repeated substitution gives:

$$\frac{\mathrm{d}V_n(\omega_n^c)}{\mathrm{d}\omega} = \frac{\beta}{\delta} - \left(\frac{\delta}{\beta}\right)^{n-1} \prod\nolimits_{j=1}^{n} g(\omega_j^c).$$

Since $g(\omega_j^c) \in [(\beta/\delta), \lambda^{-1})$, taking the limit as $n \to \infty$, or equivalently, $\omega \to \bar{\omega}$, gives $\lim_{\omega \to \bar{\omega}} \mathrm{d}V(\omega)/\mathrm{d}\omega = -\infty$. QED

Proof of Lemma 1. We establish the domain and concavity and differentiability properties of the value function $V(\omega)$.

Domain. — Since $\tau(s) \geq 0$ for all $s \in \mathcal{S}$, $\omega \geq \omega_{\min} := \sum_s \pi(s)u(e^o(s))$. The largest feasible ω, $\omega_{\max}$, can be found by solving the problem of choosing $(\tau(s), \omega'(s))$ to maximize $\sum_s \pi(s)u(e^o(s) + \tau(s))$ subject to $\tau(s) \geq 0$ and constraints (8) and (9). This is a strictly concave programming problem and the objective and constraint functions are continuous. Thus, there exists a unique solution. The constraint set is non-empty by Proposition 1. All constraints in (8) bind at the solution: if one of these constraints did not bind, say in state r, then it would be possible to increase the maximand by increasing $\tau(r)$ without violating the other constraints. Equally, it is desirable to choose $\omega'(s)$ as large as possible because an increase in $\omega'(s)$ allows $\tau(s)$ to be increased without violating constraint (8), increasing the maximand. Thus, the solution involves $\omega'(s) = \omega_{\max}$ for each s. Let $\tau^\sharp(s)$ denote the solution for the transfer and define $\omega^\sharp := \sum_s \pi(s)u(e^o(s) + \tau^\sharp(s))$. Since constraint (8) binds for each s,

$$\tau^{\sharp}(s) = e^y(s) - u^{-1}(u(e^y(s)) - \beta(\omega_{\max} - \omega_{\min})).$$

By definition $\omega_{\max} = \omega^{\sharp}$. Thus, $\omega_{\max}$ can be found as the root of

$$\sum\nolimits_s \pi(s)\left(u(e(s) - u^{-1}(u(e^y(s)) - \beta(\omega - \omega_{\min})))\right) - \omega.$$

We next show that the root $\omega_{\max} \in (\omega_{\min}, \sum_s \pi(s)u(e(s)))$. To see this, first note that $\omega_{\max} > \omega_{\min}$ by Proposition 1. For all $\omega > \omega_{\min}$, $\tau^{\sharp}(s) > 0$. Secondly, suppose that $\tau^{\sharp}(r) = e^y(r)$ for some state r. Then, $u(e^y(r)) - \beta(\omega - \omega_{\min}) = u(0)$ or

$$u(0) \geq u(e^y(r)) - \beta \sum\nolimits_s \pi(s)\left(u(e(s)) - u(e^o(s))\right),$$

which provides a contradiction since it violates Assumption 1. Hence, $\tau^{\sharp}(r) < e^y(r)$ for all r and consequently, $\omega_{\max} < \sum_s \pi(s)u(e(s))$.

Concavity. — We now show that $V(\omega)$ is concave. Consider the mapping T defined by

$$(TJ)(\omega) = \max_{\{c^y(s), \omega'(s)\} \in \Phi} \left[\sum\nolimits_s \pi(s)\left(\tfrac{\beta}{\delta}u(e(s) - c^y(s)) + u(c^y(s)) + \delta J(\omega'(s))\right)\right].$$

Consider $J = V^*$, the first-best frontier. Proposition 2 established that $V^*(\omega)$ is concave. It follows from the definitions of T and V^* that $TV^*(\omega) \leq V^*(\omega)$ because $V^*(\omega) \leq v^*/(1-\delta)$ and the mapping T adds the participation constraints (8). That is, $T^nV^*(\omega) \leq T^{n-1}V^*(\omega)$ for $n = 1$. Now, make the induction hypothesis that $T^nV^*(\omega) \leq T^{n-1}V^*(\omega)$ for $n \geq 2$ and apply the mapping T to the two functions $T^nV^*(\omega)$ and $T^{n-1}V^*(\omega)$. It is straightforward to show that $T^{n+1}V^*(\omega) \leq T^nV^*(\omega)$, because the constraint set is the same in both cases but, by the induction hypothesis, the objective is no greater in the former case. Hence, the sequence $T^nV^*(\omega)$ is non-increasing and converges. Let $V^{\infty}(\omega) = \lim_{n\to\infty} T^nV^*(\omega)$, the pointwise limit of the mapping T. We have that V^{∞} and V are both fixed points of T. Since the mapping is monotonic, $T^n(V^*) \geq T^n(V) = V$. Hence, $V^{\infty} \geq V$ but, since V is the maximum, we have that $V^{\infty} = V$. Starting from V^*, the objective function in the mapping T is concave because V^* and the utility function u are concave. The constraint set Φ is convex. Hence, $TV^*(\omega)$ is concave. By induction, $T^nV^*(\omega)$ and the limit function V are also concave.

Differentiability. — There are $2S$ choice variables and $3S + 1$ constraints, including the non-negativity constraints on transfers. Without differentiability of the value function

V, the first-order condition (13) is replaced by

$$\partial V(f(\omega, s)) \ni -\tfrac{\beta}{\delta}\left(\mu(\omega, s) - \xi(\omega, s)\right),$$

where $\partial V(\omega)$ denotes the set of superdifferentials of V at ω. Since V is concave, it is differentiable if the multipliers associated with the constraints are unique. The multipliers are unique if the linear independence constraint qualification is satisfied, that is, if the gradients of the binding constraints are linearly independent at the solution. We first note that the participation constraints of the young and the old cannot bind simultaneously in a given endowment state. If $\eta(\omega, s) > 0$, that is, the participation constraint of the old binds, then both the young and the old consume their endowments. Since the sustainable intergenerational insurance is non-autarkic, the current young receive a transfer in some endowment state when they are old and hence, they cannot be constrained in the current period, so that $\mu(\omega, s) = 0$. Likewise, if $\mu(\omega, s) > 0$, that is, the participation constraint of the young binds, then the current young are making a transfer and hence, the current old are unconstrained, so that $\eta(\omega, s) = 0$. Similarly, for $\omega < \omega_{\max}$, it is easily checked that not all upper bound constraints can bind for all states. Thus, for $\omega < \omega_{\max}$, there can be at most $2S$ binding constraints. Since utility is strictly increasing, $\beta > 0$, and $\pi(s) > 0$ for each s, it can be checked that the matrix of binding constraints has full rank. Hence, the multipliers are unique and $V(\omega)$ is differentiable on the interior of Ω. Since $V(\omega)$ is concave and differentiable, it is also continuously differentiable. It follows from the envelope condition (14) that $V_\omega(\omega_0) = 0$. Since the promise-keeping constraint (10) is an inequality, it is easily checked that $V(\omega)$ is non-increasing. The multiplier $\nu(\omega) = 0$ for $\omega < \omega_0$ and is increasing in ω for $\omega > \omega_0$. Let $\nu_{\max} := \lim_{\omega\to\omega_{\max}} \nu(\omega)$, then $\lim_{\omega\to\omega_{\max}} V_\omega(\omega) = -(\beta/\delta)\nu_{\max}$, where $\nu_{\max} \in \mathbb{R}_+ \cup \{\infty\}$.

Interiority $\omega_0 \in (\omega_{\min}, \omega^*)$. — Any non-autarkic sustainable intergenerational insurance involves some transfer from the young to the old. Thus, by Proposition 1, $\omega_0 > \omega_{\min}$. Since $V(\omega)$ is a concave Pareto frontier, that is, it is weakly decreasing and concave, it follows that $V(\omega)$ is constant for $\omega < \omega_0$ and that, by differentiability, $\nu(\omega_0) = 0$. Therefore, from the first-order condition (12), $\tau(\omega_0, s) \le \tau^*(s)$, with strict equality if the participation constraint of the young is non-binding, that is, if $\mu(\omega_0, s) = 0$. Thus, the utility promised to the old is no greater than ω^*. By Assumption 4, the first best cannot be sustained. Hence, we conclude $\omega_0 < \omega^*$. QED

C Derivation of Results of Section V

Kullback-Leibler Divergence. — The Kullback-Leibler divergence (hereafter, KL) measures the divergence between the corresponding rows of a stochastic matrix Π and a non-negative irreducible matrix Q. The two matrices are compatible if the element $\pi(x,x')=0$ whenever $q(x,x')=0$. Let $\Pi(x)$ and $Q(x)$ denote the rows of our transition matrix and state price matrix that correspond to state x. Then, the Kullback-Leibler divergence is:

$$\mathrm{KL}\left(\Pi(x)\|Q(x)\right) = -\sum\nolimits_{x'} \pi(x,x') \log\left(\frac{q(x,x')}{\pi(x,x')}\right).$$

This divergence is zero if and only if the rows are identical. By the log sum inequality, $\mathrm{KL}(\Pi(x)\|Q(x)) \geq y^1(x)$, but $y^1(x)$ could be negative if the row sum of Q corresponding to state x is greater than one. However, there is a lower bound that depends on the Perron root ρ of Q and the left eigenvector φ of Π. Define the average divergence as $\sum_x \varphi(x)\mathrm{KL}(\Pi(x)\|Q(x))$. Then,

$$\sum\nolimits_x \varphi(x)\mathrm{KL}\left(\Pi(x)\|Q(x)\right) \geq -\log(\rho).$$

Moreover, the bound is attained when

$$m(x,x') := \frac{q(x,x')}{\pi(x,x')} = \rho\frac{\psi(x)}{\psi(x')},$$

where ψ is the right eigenvector of Q corresponding to the eigenvalue ρ, that is, when the Ross Recovery Theorem holds. To see this lower bound note that the average divergence can be rewritten as:

$$\sum\nolimits_x \varphi(x)\mathrm{KL}\left(\Pi(x)\|Q(x)\right) = -\sum\nolimits_x \varphi(x)\sum\nolimits_{x'} \pi(x,x') \log\left(\frac{q(x,x')}{\pi(x,x')}\right).$$

Moreover, note that for any probability vector $\tilde{\psi}(x)$:

$$\begin{aligned}\sum\nolimits_x \varphi(x)\sum\nolimits_{x'} \pi(x,x') \log\left(\frac{\tilde{\psi}(x')}{\tilde{\psi}(x)}\right) &= \sum\nolimits_x \psi(x)\sum\nolimits_{x'} \pi(x,x') \log\left(\tilde{\psi}(x')\right) \\ &\quad - \sum\nolimits_x \varphi(x)\sum\nolimits_{x'} \pi(x,x') \log\left(\tilde{\psi}(x)\right) \\ &= \sum\nolimits_{x'} \log\left(\tilde{\psi}(x')\right)\sum\nolimits_x \varphi(x)\pi(x,x') \\ &\quad - \sum\nolimits_x \varphi(x) \log\left(\tilde{\psi}(x)\right)\sum\nolimits_{x'} \pi(x,x') = 0,\end{aligned} \tag{C.1}$$

where the last line follows because $\sum_{x'} \pi(x,x') = 1$ and $\sum_x \varphi(x)\pi(x,x') = \varphi(x')$. Hence, the left hand side of equation (C.1) is independent of $\tilde{\psi}$. Let D_ψ denote the diagonal matrix with the eigenvector ψ on the diagonal. It therefore follows from equation (C.1) that the average divergence can be rewritten as:

$$\begin{aligned}\sum\nolimits_x \varphi(x)\mathrm{KL}\left(\Pi(x)\|Q(x)\right) &= \sum\nolimits_x \varphi(x)\mathrm{KL}\left(\Pi(x)\|IQI^{-1}(x)\right)\\ &= \sum\nolimits_x \varphi(x)\mathrm{KL}\left(\Pi(x)\|\rho^{-1}D_\psi Q D_\psi^{-1}(x)\right) - \log(\rho).\end{aligned}$$

Since the matrix $\rho^{-1}D_\psi Q D_\psi^{-1}$ is stochastic and $\varphi > 0$,

$$\sum\nolimits_x \varphi(x)\mathrm{KL}\left(\Pi(x)\|\rho^{-1}D_\psi Q D_\psi^{-1}(x)\right) \geq 0,$$

with equality if $\rho^{-1}D_\psi Q D_\psi^{-1} = \Pi$.

Conditional Entropy. — Let

$$\varrho(x,x') = \frac{q(x,x')}{\sum_{x'} q(x,x')} \quad \text{and} \quad p(x) = \sum\nolimits_{x'} q(x,x'),$$

denote the risk-neutral probability of state x' when the current state is x and the price of the one period risk-free bond in state x. Let Γ denote the matrix of risk-neutral probabilities. Conditional entropy is defined by:

$$L(x) := \mathrm{KL}(\Pi(x)\|\Gamma(x)) = -\sum\nolimits_{x'} \pi(x,x')\log\left(\frac{\varrho(x,x')}{\pi(x,x')}\right).$$

Let $y(x) = -\log(p(x))$ denote the yield on the one period bond. Then,

$$L(x) = \mathrm{KL}(\Pi(x)\|\Gamma(x)) = \mathrm{KL}\left(\Pi(x)\|Q(x)\right) - y(x).$$

Since $q(x,x') = \pi(x,x')m(x,x')$, we can also write:

$$L(x) = \log\left(\sum\nolimits_{x'} \pi(x,x')m(x,x')\right) - \sum\nolimits_{x'} \pi(x,x')\log\left(m(x,x')\right).$$

Mean Entropy. — Let $\varphi(x)$ denote the probability of state x at the invariant distribution. That is φ is the left eigenvector of Π. The mean entropy is:

$$\bar{L} = \sum\nolimits_x \varphi(x)L(x).$$

Using the Ross Recovery Theorem,

$$q(x,x') = \pi(x,x')\rho\frac{\psi(x)}{\psi(x')},$$

where ρ is Perron root of Q and ψ is the corresponding right eigenvector. That is, $Q = \rho D_\psi \Pi D_\psi^{-1}$ or $\Pi = \rho^{-1} D_\psi^{-1} Q D_\psi$. Then, the bound described above is attained, and:

$$\begin{aligned}
\sum\nolimits_x \varphi(x)\mathrm{KL}\left(\Pi(x)\|Q(x)\right) &= -\sum\nolimits_x \varphi(x)\sum\nolimits_{x'} \pi(x,x')\log\left(\frac{q(x,x')}{\pi(x,x')}\right) \\
&= -\log(\rho) - \sum\nolimits_{x'}\Big(\varphi(x') - \sum\nolimits_x \varphi(x)\pi(x,x')\Big)\log\left(\psi(x')\right) \\
&= -\log(\rho),
\end{aligned}$$

where the last line follows because $\sum_x \varphi(x)\pi(x,x') = \varphi(x')$. Hence,

$$\bar{L} = \sum\nolimits_x \varphi(x)L(x) = -\log(\rho) - \sum\nolimits_x \varphi(x)y(x).$$

Since $y^\infty = -\log(\rho)$ and letting $\bar{y} = \sum_x \varphi(x)y(x)$ denote the average yield, we have:

$$\bar{L} = \sum\nolimits_x \varphi(x)L(x) = y^\infty - \bar{y}.$$

Repeating for the k-period entropy gives equation (19) in the text. Note that:

$$\bar{L} = \sum\nolimits_x \varphi(x)\mathrm{KL}\left(\Pi(x)\|\Gamma(x)\right) = \sum\nolimits_x \varphi(x)\left(\mathrm{KL}\left(\Pi(x)\|Q(x)\right) - y(x)\right),$$

which can be rewritten using $\bar{L} = y^\infty - \bar{y}$ as:

$$-\log(\rho) = \sum\nolimits_x \varphi(x)\left(y(x) + \mathrm{KL}\left(\Pi(x)\|\Gamma(x)\right)\right) = \bar{y} + \bar{L}.$$

Martin-Ross Measure. — Let $\psi_{\max} = \max_x \psi(x)$ and $\psi_{\min} = \min_x \psi(x)$. Define the Martin-Ross measure:

$$\Upsilon := \log\left(\frac{\psi_{\max}}{\psi_{\min}}\right).$$

It follows from the Ross Recovery Theorem that for each pair (x, x'):

$$\log\left(m(x,x')\right) - \log(\rho) = \log\left(\frac{\psi(x)}{\psi(x')}\right),$$

and hence, using the definitions of $\psi_{\max}$ and $\psi_{\min}$,

$$-\Upsilon \le \log\left(m(x,x')\right) - \log(\rho) \le \Upsilon.$$

Since ψ is the corresponding eigenvector, we have the following two sets of inequalities:

$$\rho\psi_{\min} \leq \rho\psi(x) = \sum\nolimits_{x'} q(x,x')\psi(x') \leq \sum\nolimits_{x'} q(x,x')\psi_{\max} = p(x)\psi_{\max},$$
$$\rho\psi_{\max} \geq \rho\psi(x) = \sum\nolimits_{x'} q(x,x')\psi(x') \geq \sum\nolimits_{x'} q(x,x')\psi_{\min} = p(x)\psi_{\min}.$$

Taking logs and using $\log(\rho) = -y^{\infty}$, $|y(x) - y^{\infty}| \leq \Upsilon$. Since $\bar{L} = y^{\infty} - \bar{y}$, it follows that $\bar{L} - \Upsilon \leq y(x) - \bar{y} \leq \bar{L} + \Upsilon$. Repeating for the k-period case, gives:

$$\frac{\bar{L} - \Upsilon}{k} \leq y^k(x) - \bar{y}^k \leq \frac{\bar{L} + \Upsilon}{k}.$$

D Shooting Algorithm

In the two-state economy in Section VI, the multiplier on the participation constraint in state 2 satisfies $\mu(\omega, 2) = 0$ for all $\omega \in [\omega_0, \bar{\omega}]$. Therefore, write $\upsilon(\omega) := 1 + \mu(\omega, 1)$. At the invariant distribution, write $\upsilon^{(n)} = \upsilon(\omega^{(n)})$ where $\omega^{(n)}$ is the promised utility after n consecutive state 1s. Using the updating property of equation (15), $\nu(\omega^{(n+1)}) = \mu(\omega^{(n)})$ and equation (12) can be written as:

$$\frac{e(1) - c^y(\omega^{(n)}, 1)}{c^y(\omega^{(n)}, 1)} = \frac{\beta}{\delta}\left(\frac{\upsilon^{(n-1)}}{\upsilon^{(n)}}\right),$$
$$\frac{e(2) - c^y(\omega^{(n)}, 2)}{c^y(\omega^{(n)}, 2)} = \frac{\beta}{\delta}\upsilon^{(n-1)}.$$

Given Assumptions 3 and 4, both the participation constraint of the young in state 1 and the promise-keeping constraint bind. That is,

$$\pi \log\left(\tfrac{\beta\upsilon^{(n-1)}e(1)}{\beta\upsilon^{(n-1)}+\delta\upsilon^{(n)}}\right) + (1-\pi)\log\left(\tfrac{\beta\upsilon^{(n-1)}e(2)}{\beta\upsilon^{(n-1)}+\delta}\right) = \omega^{(n)}, \tag{D.1}$$
$$\log\left(\tfrac{\delta\upsilon^{(n)}e(1)}{\beta\upsilon^{(n-1)}+\delta\upsilon^{(n)}}\right) + \beta\omega^{(n+1)} = \log(e^y(1)) + \beta\omega_{\min}, \tag{D.2}$$

for $n \geq 0$ where $\upsilon^{(-1)} = 1$. For $n = 0$,

$$\pi \log\left(\tfrac{\beta e(1)}{\beta+\delta\upsilon^{(0)}}\right) + (1-\pi)\log\left(\tfrac{\beta e(2)}{\beta+\delta}\right) = \omega_0,$$

while for n that tends to infinity,

$$\pi \log\left(\tfrac{\beta e(1)}{\beta+\delta}\right) + (1-\pi)\log\left(\tfrac{\beta\upsilon^{(\infty)}e(2)}{\beta\upsilon^{(\infty)}+\delta}\right) = \bar{\omega}, \tag{D.3}$$

where $\upsilon^{(\infty)} = \lim_{n\to\infty} \upsilon^{(n)}$ and

$$\bar{\omega} = \frac{1}{\beta}\left(\log(e^y(1)) - \log\left(\tfrac{\beta e(1)}{\beta+\delta}\right)\right) + \pi\log(e^o(1)) + (1-\pi)\log(e^o(2)). \tag{D.4}$$

Substituting equation (D.4) into (D.3), we have:

$$\upsilon^{(\infty)} = \frac{\delta}{\beta}\left(-1+\left(\left(\tfrac{\delta}{\beta}\right)^{\frac{\pi}{1-\pi}}\left(\tfrac{\beta+\delta}{\delta}\right)^{\frac{1+\beta\pi}{\beta(1-\pi)}}\left(\tfrac{e^y(1)}{e(1)}\right)^{\frac{1}{\beta(1-\pi)}}\left(\tfrac{e^o(1)}{e(1)}\right)^{\frac{\pi}{1-\pi}}\tfrac{e^o(2)}{e(2)}\right)^{-1}\right)^{-1}. \tag{D.5}$$

Using the equations (D.1) and (D.2), we can derive a second-order difference equation for $\upsilon^{(n)}$ where

$$\upsilon^{(n+1)} = \tfrac{\beta}{\delta}\upsilon^{(n)}\left(-1+\left(\tfrac{\beta\upsilon^{(n)}}{\beta\upsilon^{(n)}+\delta}\right)^{\frac{1-\pi}{\pi}}\left(\tfrac{\beta\upsilon^{(\infty)}+\delta}{\beta\upsilon^{(\infty)}}\right)^{\frac{1-\pi}{\pi}}\left(\tfrac{\beta+\delta}{\delta}\right)^{\frac{1}{\beta\pi}}\left(\tfrac{\beta+\delta}{\beta}\right)\left(1+\tfrac{\beta}{\delta}\tfrac{\upsilon^{(n-1)}}{\upsilon^{(n)}}\right)^{-\frac{1}{\beta\pi}}\right). \tag{D.6}$$

It can be shown that the second-order difference equation in (D.6) has a unique saddle path solution. Recalling that $\upsilon^{(-1)} = 1$, the solution can be found by a *forward shooting* algorithm to search for an $\upsilon^{(0)}$ such that the absolute difference between $\upsilon^{(\infty)}$ (given in (D.5)) and $\upsilon^{(N+1)}$ (given in (D.6)) is sufficiently close to zero for N sufficiently large.

E Comparative Statics

In this section, we continue the two-state example of Section VI and examine how the generational risk measures and the autocorrelation of consumption of the young across two adjacent generations respond to comparative static changes of endowment parameters and discount factors.[45] For all comparative statics, we change the value of the parameter of interest holding all other parameters at the values in the canonical case of Example 1.[46]

Changing the Endowment. — The effect of changes in κ and σ are illustrated in the first two columns of Figure E.1. The top row illustrates the effect on $\bar{L}$ and $\bar{L}\pm\Upsilon$ and the bottom row illustrates the effect on the autocorrelation of consumption. A larger κ corresponds to a larger average endowment share to the young, while a smaller σ corresponds to reduced idiosyncratic uncertainty. Increasing κ, or reducing σ, increases

[45] For the purposes of comparison, we use autocorrelation instead of auto-covariance. The conditional autocorrelation is given by $\operatorname{corr}(c_t^y, c_{t+1}^y \mid s_t) := \operatorname{cov}(c_t^y, c_{t+1}^y \mid s_t)/\sqrt{(\operatorname{var}(c_t^y \mid s_t)\operatorname{var}(c_{t+1}^y \mid s_t))}$ and the unconditional autocorrelation is given by $\operatorname{corr}(c_t^y, c_{t+1}^y) := \operatorname{cov}(c_t^y, c_{t+1}^y)/\operatorname{var}(c_t^y)$.

[46] In all cases, the invariant distribution is geometric, except when discount factors are changed. When the invariant distribution is not geometric, we can no longer rely on the shooting algorithm used in Section VI. In this case, we implement an algorithm based on a value function iteration method (see, Part H of the Supplementary Appendix for a description). Although we consider an example with two endowment states here, the value function iteration method can be applied when there are more than two states.

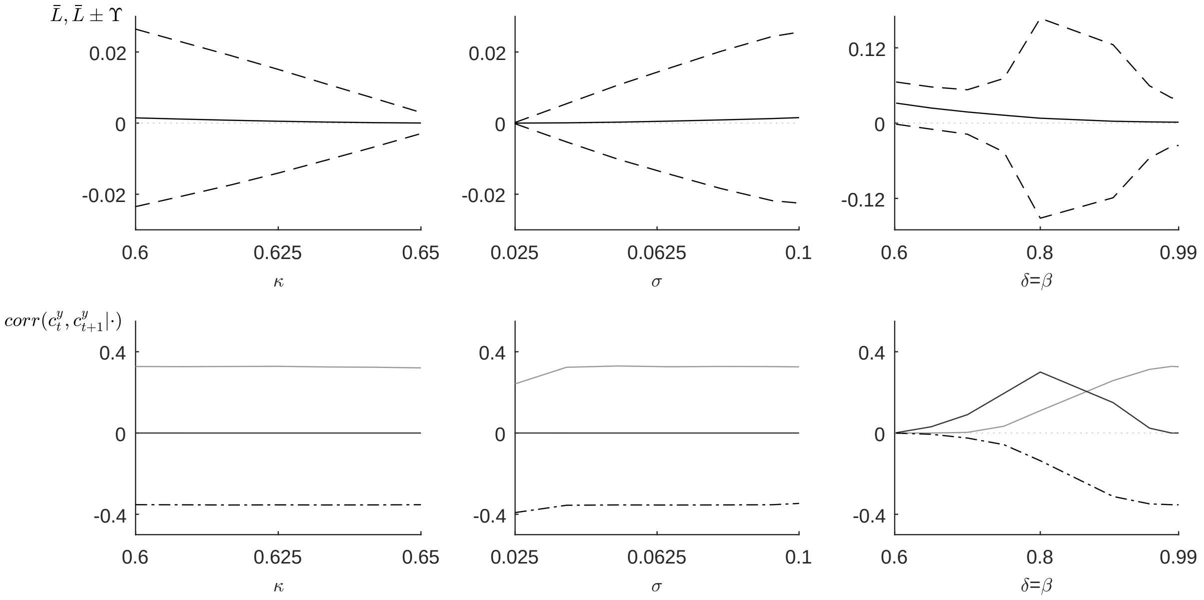

Figure E.1: Comparative Statics at the Invariant Distribution.

Note. — In the top row, the solid line is $\bar{L}$ and the dashed lines are $\bar{L} \pm \Upsilon$ (where Υ is scaled by 1/10). In the bottom row, the light gray line is the autocorrelation of consumption of the young between adjacent periods conditional on state 1, $\text{corr}(c_t^y, c_{t+1}^y \mid s_t = 1)$, the dark gray line is the corresponding autocorrelation conditional on state 2, $\text{corr}(c_t^y, c_{t+1}^y \mid s_t = 2)$, and the dashed line is the unconditional autocorrelation, $\text{corr}(c_t^y, c_{t+1}^y)$.

risk sharing as measured by a reduction in $\bar{L}$ and Υ. For κ above some critical value, or σ below a critical value, the first best is sustainable in the long run at the invariant distribution, in which case $\bar{L} = \Upsilon = 0$.[47] For the range of κ and σ illustrated, the premise of Propositions 6 and 7 hold, that is, $f(\omega, 2) = \omega_0$ for $\omega \leq \bar{\omega}$ and $\bar{\omega} < \omega_{\max}$. The implications of this premise can be seen in the bottom row of Figure E.1, where the autocorrelation of consumption conditional on state 2 is zero. The corresponding autocorrelation conditional on state 1 is positive, while the unconditional autocorrelation is negative. Neither is very sensitive to changes in κ and σ. Although the auto-covariance of consumption tends to zero as the consumption tends to the first best (for κ large enough, or σ small enough), the variance of consumption decreases at a similar rate, so that there is little change in the autocorrelation coefficient.

Changing the Discount Factors. — The final column of Figure E.1 illustrates the effect of changes in the discount factor (holding $\beta = \delta$). The effect on the generational risk measures and the autocorrelation of consumption is non-monotonic. This occurs because the assumptions $f(\omega, 2) = \omega_0$ for $\omega \leq \bar{\omega}$ and $\omega < \omega_{\max}$ do not hold when the discount factor is sufficiently small. For high values of the discount factor, the invariant distribution

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[47] The critical values are $\kappa \approx 0.6565$ and $\sigma \approx 0.0243$.

has geometric probabilities as described in part (i) of Proposition 7. As the discount factor is decreased, either the current transfer is reduced, or the future promise increased, to satisfy the participation constraint of the young in state 1. This change spreads out the distribution of ω, increasing $\bar{\omega}$ but reducing ω_0 and $\omega_{\max}$. The reduced risk sharing and the increased spread of promised utility are reflected in an increase of $\bar{L}$ and Υ. The variance of consumption is increased, but so too is the absolute value of the auto-covariance of consumption, with an overall reduction in the absolute value of the autocorrelation (unconditional as well as conditional on state 1). As the discount factor is reduced further, both the upper bound constraint becomes binding, that is, $\bar{\omega} = \omega_{\max}$, and $f(\omega, 2) > \omega_0$ for high values of ω. Reversion to ω_0 occurs less frequently and as the discount factor falls, the invariant distribution has a positive probability mass at $\omega_{\max}$. Although ω_0 falls with the discount factor, the range $\bar{\omega} - \omega_0$ decreases, meaning that although $\bar{L}$ increases, the bound Υ decreases. For high values of ω, the future promise is strictly increasing in ω even in state 2 and therefore, the autocorrelation of consumption conditional on state 2 is positive. For a low enough discount factor ($\beta = 3/5$ in the figure), autarky is the only sustainable allocation: the invariant distribution tends to a degenerate distribution with a unit mass on $\omega_{\min}$ and the autocorrelation of consumption approaches zero because the endowments are serially uncorrelated.

F Alternative Measures of Risk Sharing

In this Appendix, we compute the insurance coefficient and the consumption equivalent welfare change at the invariant distribution in the two-state example of Section VI. We do this for the range of parameter values considered in Part E of this Appendix.

The insurance coefficient $\iota(x)$ is the fraction of the variance of the endowment shock that does not translate into a corresponding change in consumption. With i.i.d. shocks, it is defined conditional on state $x = (\omega, s)$ as follows:

$$\iota(x) = 1 - \frac{\operatorname{cov}\left(\log\left(c^y(f(x), s')\right), \log\left(e^y(s')\right)\right)}{\operatorname{var}\left(\log\left(e^y(s')\right)\right)}.$$

At the first best, and provided that the non-negativity constraint does not bind, consumption is independent of the endowment and the insurance coefficient is one. The insurance coefficient increases as more risk is shared. With two states, the insurance coefficient can be rewritten as:

$$\iota(x) = 1 - \frac{\log\left(c^y(f(x), 1)\right) - \log\left(c^y(f(x), 2)\right)}{\log\left(e^y(1)\right) - \log\left(e^y(2)\right)}.$$

The top row of Figure F.1 plots the average insurance coefficient evaluated at the invariant distribution of x for the three parameters κ, σ and δ when $\beta = \delta$.

We measure the consumption equivalent welfare change relative to the first best for a given ω by solving the following equation in terms of ε:

$$\frac{1}{1-\delta}\left(\mathbb{E}_s[u(c^{y*}(s)(1-\varepsilon))] + \frac{\beta}{\delta}\mathbb{E}_s[u((e(s) - c^{y*}(s))(1-\varepsilon))]\right) = V(\omega).$$

The solution $\varepsilon(\omega)$ measures the proportion by which the first-best consumption needs to be reduced to match the optimal solution for each ω. The consumption equivalent welfare change is smaller when more risk is shared. The long-run welfare loss measure is the average of $\varepsilon(\omega)$ at the invariant distribution of ω. The bottom row of Figure F.1 plots the consumption equivalent welfare change for the three parameters κ, σ and δ when $\beta = \delta$.

Comparing Figure F.1 with the mean entropy measure illustrated in top row of Figure E.1, it can be seen that broadly similar conclusions are obtained using mean entropy, the average insurance coefficient or the average consumption equivalent welfare change. The amount of risk shared at the optimal sustainable intergenerational insurances increases with κ and δ but falls with σ.

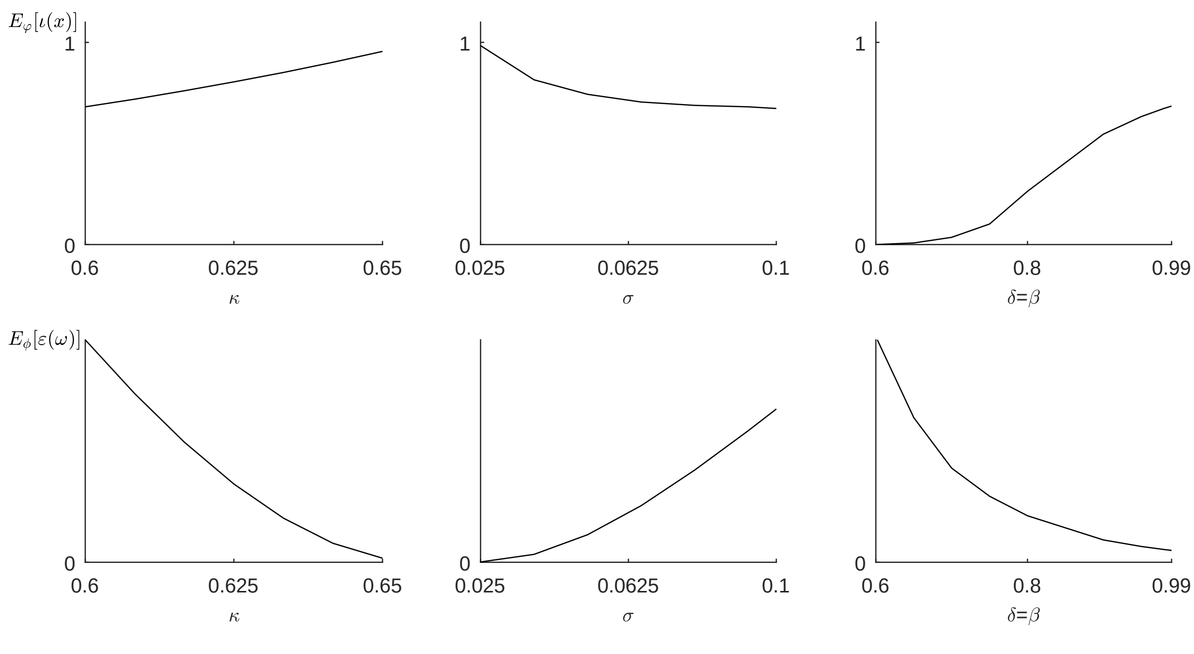

Figure F.1: Insurance Coefficient and the Consumption Equivalent Welfare Change

Note. — The top row illustrates the average insurance coefficient $\mathbb{E}_\varphi[\iota(x)]$. The bottom row illustrates the average consumption equivalent welfare change $\mathbb{E}_\phi[\varepsilon(\omega)]$.

G Extensions

This appendix provides some further details of the extensions of the basic model that are discussed in Section VIII.

Heterogeneous Preferences. — The assumption in the basic model that the young and the old have a common utility function is not essential. Allowing for heterogeneity of preferences complicates the notation but does not change the results. As a special case, consider the situation in which the young have risk-neutral preferences. In this case, the ordering of the states $\lambda(s) \geq \lambda(q)$ for $s > q$ implies that the endowment of the old is monotonic in s: $e^o(s) \geq e^o(q)$. It can then be verified that the policy functions for the future promised utility and the consumption of the old are also monotonic in s: $f(\omega, s) \leq f(\omega, q)$ and $c^o(\omega, s) \geq c^o(\omega, q)$ for $s > q$.

Growth. — It has been assumed that the distribution of endowments is the same at all dates. This can be generalized to allow for stochastic growth of the endowment. To do this, decompose the endowment state into an idiosyncratic shock θ and an aggregate growth shock Θ so that $s = (\theta, \Theta)$. Suppose that the idiosyncratic and aggregate components are i.i.d. and independent of each other, so that $\pi(s) = \pi(\theta)\pi(\Theta)$ where $\pi(\theta)$ and $\pi(\Theta)$ are the probabilities of the two shocks. Let $\alpha(\theta) = e^y_t/e_t$ denote the endowment share of the young and suppose that it depends only on θ. Similarly, let $\chi(\Theta) = e_t/e_{t-1}$ denote the growth rate of the aggregate endowment and suppose that it depends only on Θ at time t. Furthermore, suppose that the preferences of both agents exhibit CRRA with relative risk aversion coefficient γ. It is then possible to rewrite the planner's problem to be stationary as follows: normalize consumption at t by dividing by e_t, normalize utility variables ω, V and $\hat{v}$, at t by dividing by $e_t^{1-\gamma}$ and normalize the discount factors by multiplying by $\sum_\Theta \pi(\Theta)\chi(\Theta)^{1-\gamma}$. The modified planner's problem with these normalized variables is identical to the stationary problem described in Section III.[48] Hence, the solution with a stochastic growth component is obtained by a simple reinterpretation of the variables.

Savings. — We have assumed that the only method of insurance is through intergenerational transfers. Now suppose that the young have access to a linear storage technology that delivers R units of endowment when they are old for every unit stored. Suppose that access to storage is not available in autarky but only to the young who do not default on the transfers they are called on to make. It is clear that storage is not used if the gross rate of return R is too low. In other words, there is an $\bar{R}$ such that for $R < \bar{R}$, storage is not used even if available. Given the solution to the optimal intergenerational insurance

[48] The growth rate cannot be too high. The modified discount factor must satisfy $\delta \sum_\Theta \pi(\Theta)\chi(\Theta)^{1-\gamma} < 1$ for the planner's problem to be finite.

rule, $\bar{R}^{-1} = \max_x\{p^1(x)\}$. In the two-state case of Section VI, the maximum is attained when $x = (\omega_0, 1)$. With the parameter values of Example 1, $\bar{R} \approx 1.08171$, demonstrating that even when storage has a positive net return, the possibility of storage may have no impact on the optimal sustainable intergenerational insurance.

The situation is only slightly different if individuals have access to the same storage technology, with gross rate of return R, in autarky. Let a denote the amount stored, then the lifetime autarkic utility is:

$$\hat{v}(s;R) := \max_{a\geq 0}\, u(e^y(s)-a) + \beta \sum\nolimits_{s'} \pi(s')u(e^o(s') + Ra),$$

Let $a(s;R)$ denote the optimal amount stored. The function $\hat{v}(s;R)$ is increasing in R and $a(s;R)$ is weakly increasing in R (strictly if $a(s;R) > 0$). Hence, there is a critical value $\hat{R}$ such that for $R < \hat{R}$, $a(s;R) = 0$ for each s. In particular, $\hat{R}^{-1} = \max_s \sum_{s'} \hat{q}(s,s')$, where $\hat{q}(s,s')$ is the state price evaluated in autarky. We can show that $\hat{R} < 1$. To see this, note that Assumption 2 requires that the Perron root of the matrix $\hat{Q}$ is greater than one. This can only occur if at least one of the row sums of $\hat{Q}$ is greater than one: $\sum_{s'} \hat{q}(s,s') > 1$ for some state s (or equivalently, $\hat{R} < 1$).[49] Nevertheless, for $R \leq \hat{R}$, there is no storage in autarky and provided that $R \leq \min\{\hat{R}, \bar{R}\}$, the possibility of storage has no effect on the optimal sustainable intergenerational insurance.

Altruism. — To incorporate altruism into the model, we consider "warm glow" preferences where the old attach a weight of $\varsigma > 0$ to the utility of the young. In this case, the lifetime utility of an individual born after the history s^t is:

$$u(c^y(s^t)) + \beta \sum\nolimits_{s_{t+1}} \pi\left(s_{t+1}\right)\left(u(c^o(s^{t+1})) + \varsigma u(c^y(s^{t+1}))\right).$$

With these preferences, the participation constraints of the old and the young are:

$$u\left(e(s) - c^y(s)\right) + \varsigma u\left(c^y(s)\right) \geq u(e^o(s)) + \varsigma u(e^y(s)) \quad \forall\ s \in \mathcal{S},$$
$$u\left(c^y(s)\right) + \beta\omega'(s) \geq u\left(e^y(s)\right) + \beta \sum\nolimits_{s'} \pi\left(s'\right)\left(u(e^o(s')) + \varsigma u(e^y(s'))\right) \quad \forall\ s \in \mathcal{S},$$

and the promise-keeping constraint is:

$$\sum\nolimits_{s} \pi\left(s\right)\left(u\left(e\left(s\right) - c^y\left(s\right)\right) + \varsigma u\left(c^y\left(s\right)\right)\right) \geq \omega.$$

[49] For the two-state case of Section VI, and setting $\gamma = 1$ and $\pi = 1/2$, it can be verified that $\hat{R} = \beta^{-1}((1-\kappa)^2 - \sigma^2)/((1-\kappa)(\kappa+\sigma))$. Assumption 2 is satisfied for $\beta \geq ((1-\kappa)^2 - \sigma^2)/(\kappa(1-\kappa)+\sigma^2))$. Hence, $\hat{R} \leq (\kappa(1-\kappa)+\sigma^2)/((1-\kappa)(\kappa+\sigma)) < 1$. With $\kappa = 3/5$ and $\sigma = 1/10$, we have $\hat{R} \leq 25/28$.

The analysis of Section III can be applied *mutatis mutandis*. The first-order conditions for the optimal sustainable intergenerational insurance are given by equation (13) and

$$\frac{u_c\left(c^y(x)\right)}{u_c\left(e(s)-c^y(x)\right)}=\frac{\beta}{\delta}\left(\frac{1+\nu(\omega)+\eta\left(x\right)}{1+\mu(x)+\varsigma\frac{\beta}{\delta}\left(1+\nu(\omega)+\eta\left(x\right)\right)}\right), \tag{G.1}$$

where multipliers are as specified in Section III. Condition (G.1) reduces to condition (12) when $\varsigma=0$. Lemmas 2 and 3 continue to hold provided that ς is not too large.

Markov Endowments. — In the basic model, endowments are i.i.d. To see how the i.i.d. assumption can be generalized, we consider the two-state example of Section VI and its invariant distribution. Let the probability of remaining in endowment state 1 be $\pi+(1-\pi)r$ and the probability of moving from endowment state 2 to endowment state 1 be $\pi(1-r)$, where r is a persistence parameter. The i.i.d. case corresponds to $r=0$. If $r>0$, then the probability of remaining in state 1 is increased and the probability of moving from state 2 to state 1 is reduced compared to the i.i.d. case.[50] With this parameterization, the long-run probabilities of state 1 and state 2 occurring are π and $1-\pi$ respectively.[51] From Section VI, there are parameter values such that the young are never constrained in state 2. By continuity the same property will apply provided r is close to zero. In that case, the optimum will depend on the number of consecutive state 1s in the most recent history and the shooting algorithm discussed in Section VI can be adapted by using the conditional probabilities for state 1.[52] At the invariant distribution, the probability of state 2 is $(1-\pi)$, the probability of one consecutive state 1 is $(1-\pi)\pi(1-r)$, the probability of two consecutive state 1s is $(1-\pi)\pi(1-r)(\pi+(1-\pi)r)$, and so on. Hence, $\phi(\{\omega^{(0)}\})=1-\pi$ and $\phi(\{\omega^{(n)}\})=(1-\pi)\pi(1-r)(\pi+(1-\pi)r)^{n-1}$ for $n=1,2,\ldots,\infty$, where each $\omega^{(n)}$ is computed from the solution to the modified shooting algorithm. For $r=0$, this reduces to the result given in Part (i) of Proposition 7.

H Pseudo-code for Numerical Algorithms

Algorithms are implemented in MATLAB®. At each iteration, the optimization uses the nonlinear programming solver command `fsolve` in Algorithm 1 and command `fmincon`

[50] If $r=1$, then we have the deterministic case examined in Section II. It is also possible to consider cases where $r<0$ provided that $r>\max\{-\pi/(1-\pi),-(1-\pi)/\pi\}$.

[51] If the initial probability of endowment state 1 is π, then these long-run probabilities hold at every date and r is the correlation coefficient of the endowments at two consecutive dates.

[52] The probability of remaining in state 1 is $\pi+(1-\pi)r$ and the probability of moving from state 1 to state 2 is $(1-\pi)(1-r)$. One further assumption is required to apply the shooting algorithm. It should be optimal to set consumption to the first-best level in state 2 whenever there are no immediately proceeding state 1s (as in the i.i.d. case). This occurs whenever the initial ω is not too high, a property that can easily be checked once the solution is computed.

in Algorithm 2. Value function interpolation uses the spline method of the `interp1` command. In a typical example, the value function converges within 300 iterations.

Algorithm 1: Shooting Algorithm

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procedure ▷ Find $\upsilon^{(0)} = 1 + \mu^{(0)}$ in two state economy (Section VI)
  target $\leftarrow \upsilon^{(\infty)}$ ▷ Use equation (D.5) in Appendix D
  tolerance $\leftarrow \epsilon > 0$ ▷ $\epsilon = 10^{-10}$
  repeat
    initialization $\leftarrow \upsilon_0^{(0)} > 0$
    Compute $\upsilon_0^{(N)}$ for $N = 20$ ▷ Use equation (D.6) in Appendix D
    $d \leftarrow d(\upsilon_0^{(N)}, \upsilon^{(\infty)})$ ▷ $d(\upsilon_0^{(N)}, \upsilon^{(\infty)}) = |\upsilon_0^{(N)} - \upsilon^{(\infty)}|$
  until $d < \epsilon$
  $\upsilon^{(0)} \leftarrow \upsilon_0^{(0)}$
end procedure

Algorithm 2: Find Value and Policy Functions

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procedure ▷ Find solution to functional equation (11)
  $\Omega \leftarrow [\omega_{\min}, \bar{\omega}]$ ▷ $\omega_{\min}$ and $\bar{\omega}$ computed
  gridpoints $\leftarrow gp$ ▷ Discretize Ω: $gp = 200$ Chebyshev interpolation points
  tolerance $\leftarrow \epsilon > 0$ ▷ $\epsilon = 10^{-6}$
  $J \leftarrow V^*$ ▷ V^* is first best
  repeat
    Compute TJ from J ▷ Use equation (11) and interpolate
    $d \leftarrow d(TJ, J)$ ▷ $d(TJ, J) = \max_\omega |TJ(\omega) - J(\omega)|$
    $J \leftarrow TJ$
  until $d < \epsilon$
  $V \leftarrow J$
  Compute $f(\omega, s)$ and $c^y(\omega, s)$ ▷ Using the function V just computed.
end procedure

Algorithm 3: Computing the Invariant Distribution

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procedure ▷ Find invariant distribution for $x = (\omega, s) \in X \subset \mathbb{R}^{nS\times 1}$
  initialization $\leftarrow a_0 = \mathbf{e}(1/nS)$ ▷ $\mathbf{e} = (1, 1, ..., 1) \in \mathbb{R}^{nS\times 1}$
  Compute $a = \Pi a_0$ ▷ Use the transition probability $\Pi \subset \mathbb{R}^{nS\times nS}$
  tolerance $\leftarrow \epsilon > 0$ ▷ $\epsilon = 10^{-8}$
  repeat
    Compute $a = \Pi a$ ▷ a is eigenvector associated with 1
    $d \leftarrow d(\Pi a, a)$ ▷ $d(\Pi a, a) = \max_x |\Pi a(x) - a(x)|$
    $a \leftarrow \Pi a$
  until $d < \epsilon$
  $\varphi \leftarrow a / \sum_x a(x)$ ▷ φ is normalized invariant distribution
end procedure
