Ramsey Theory

Ramsey Theory of Taxation

- Originated in the work of Frank Ramsey in the 1920's
- Developed in public finance in 1960's, 1970's
- Applied to macroeconomics from 1980's onwards

Basic Assumptions

- Benevolent social planner (SP) endowed with social preferences
- Most Ramsey theory assumes agents are all the same type
- SP must use **linear taxes** to finance public exogenous expenditure. It cannot use a lump sum tax (i.e. demand same tax payment from everyone)
- Underlying motivation: people are different, their types are unobservable, linear taxes fairer than lump sum
- Multiple instruments: Tax on labor income, capital income, consumption

Roadmap

- Static Problem: Optimal Commodity Taxation (Atkinson and Stiglitz, 1976)
- Dynamic Problem: Optimal Income (labor and capital) Taxation
  - Without Uncertainty (Judd, 1985; Chamley, 1986)
  - With Uncertainty and Complete Market:

    Without Capital (Lucas and Stokey, 1983)

    With Capital (Chari, Christiano, and Kehoe, 1994)
  - With Uncertainty and Incomplete Market (Aiyagari, Marcet, Sargent, and Seppala, 2002) $\Rightarrow$ Barro (1979)

Preview of Main Results

- Tax distortions should be smoothed over time and over states of nature
- Capital income taxes: High initially, then roughly zero
- Tax rates on labor and consumption income should be stable over time
- State contingent taxes on assets/ state contingent debt should be use to provide insurance against shocks

Static Problem:
Optimal Commodity Taxation

Optimal Commodity Taxation: Atkinson-Stiglitz, 1976

- **Static economy** and **general equilibrium** setup
- n different consumption goods $i = 1, ..., n$
- Large population of identical agents
- Each agent consumes n different consumption goods and supplies labor l
- A large number of firms operate identical, constant return to scale technology to produce consumption goods
- SP relies on commodity taxes to finance exogenous expenditure $\{g_i\}$ of each good.

Environment

- Treat labor as numeraire (wage rate is equal to 1) and denote:
  - p_i: price of i-th good
  - τ_i: tax on consumption of i-th good
- Agents solves:

$$\max\ U(c_1, ..., c_n, l) \quad \text{s.t.} \quad \sum_i p_i(1+\tau_i)c_i = l$$

- Firms solves:

$$\max \sum_i p_i y_i - l \qquad \text{s.t.} \quad F(y_1, ..., y_n, l) = 0$$

- SP's budget constraint:

$$\sum_i p_i g_i = \sum_i p_i \tau_i c_i$$

- Market clearing condition: $c_i + g_i = y_i \quad \forall_i$

Competitive Equilibrium

Definition: With taxes $\{\tau_i\}$ and government purchases $\{g_i\}$, allocations $\{y_i, c_i, l\}$ and prices $\{p_i\}$ form a **competitive equilibrium** if and only if:

- Taking $\{p_i\}$ as given, $\{c_i, l\}$ solve consumer's problem
- Taking $\{p_i\}$ as given, $\{y_i, l\}$ solve firm's problem
- The government budget constraint holds
- Goods markets clear

Question

How to find $\{\tau_i\}$ to finance public expenditure $\{g_i\}$ so that welfare is maximized?

- Two different approaches:
    1. Express everything as a function of τ and maximize w.r.t. τ directly
    2. Use **primal** (Ramsey) approach
- We will take the second approach

Idea: Find necessary and sufficient conditions on $\{c_i, l_i\}$ that should be true in any CE, and then find the $\{c_i, l_i\}$ that satisfy these conditions and maximize the welfare

Implementability Constraint

- Consumer's FOCs:

$$U_{c_i} = \lambda p_i(1+\tau_i) \quad \text{and} \quad U_l = -\lambda$$

  where λ is the Lagrange multiplier on the budget constraint

- Combining Focs implies

$$p_i(1+\tau_i) = -\frac{U_{c_i}}{U_l}$$

  and substituting back into consumer's budget constraint yields

$$\sum_i U_{c_i} c_i + U_l l = 0 \tag{ic}$$

- Eq. (ic) is called **implementability constraint**

Characterize Competitive Equilibrium

Proposition: For any exogenous stream $(g_1, ..., g_n)$, an interior allocation $(c_1^*, ..., c_n^*, l^*)$ is an interior competitive allocation if and only if:

$$F(c_1 + g_1, ..., c_n + g_n, l) = 0 \qquad \text{(rc)}$$

and

$$\sum U_{c_i} c_i + U_l l = 0$$

⇒ Then there exists a competitive equilibrium with taxes for which $(c_1^*, ..., c_n^*, l^*)$ are equilibrium allocations

- This may seem a little surprising since we have $n + 1$ variables and only two constraints
  - This means that there exist many solutions to this system of equations. For any solution that satisfies these conditions, we can find some taxes that would implement them

Re-constructing Equilibrium from Allocations

- Pick any $(c_1^*, ..., c_n^*, l^*)$ that satisfies conditions (rc) and (ic)
- **Construct prices**: from firm's problem we have

$$p_i = \lambda F_i \quad \text{and} \quad -1 = \lambda F_l$$

- Therefore, combining yields

$$p_i = -\frac{F_i(c_1^* + g_1^*, ..., c_n^* + g_n^*, l^*)}{F_l(c_1^* + g_1^*, ..., c_n^* + g_n^*, l^*)}$$

- **Construct taxes**: from consumer's Focs

$$1+\tau_i = -\frac{\frac{U_{c_i}(c_1^*, ..., c_n^*, l^*)}{U_l(c_1^*, ..., c_n^*, l^*)}}{p_i} = \frac{U_{c_i}(c_1^*, ..., c_n^*, l^*)}{U_l(c_1^*, ..., c_n^*, l^*)} \frac{F_l(c_1^* + g_1^*, ..., c_n^* + g_n^*, l^*)}{F_i(c_1^* + g_1^*, ..., c_n^* + g_n^*, l^*)}$$

Remaining Sufficiency Conditions

- Are Firms making zero profit? Yes, since F is CRS

$$\sum F_i c_i^* + F_l l^* = 0$$

- Does it raise enough money to finance the government?

$$\sum_i p_i g_i = \sum_i p_i \tau_i c_i$$

  - Substitute definition of prices, taxes and consumer budget constraint to verify that it holds
  - Also follows from Walras' law

Primal Approach

- We have the following feasible set of allocation:

$$\Delta = \{(c, l) \in R^n_+ \times (0, \bar{l}) \mid \text{(rc) and (ic)}\}$$

  - This leaves out some competitive allocations in which $l = 0$, $l = \bar{l}$ or $c_i = 0$
- How to find the allocation that maximizes social welfare?
- **Policy design problem**:

$$\max_{\Delta} U(c_1, ..., c_n, l)$$

  - Solve. Then recover supporting taxes
- So far, only taxes on consumption goods, none on labor income
  - Could have taxes on labor income as well. Then $n + 1$ taxes. Only need n of these

Solving the Design Problem

- For simplicity, suppose U is additively separable:

$$U(c_1, ..., c_n, l) = \sum_{i=1}^{n} u_i(c_i) + v(l)$$

- FOCs w.r.t. c_i and l are:

$$(1+\lambda)u_{c_i} + \lambda u_{c_i c_i} c_i = \gamma F_i$$

and

$$(1+\lambda)v_l + \lambda v_{ll} l = \gamma F_l$$

where λ and γ are the Lagrange multipliers on (ic) and (rc)

- Combining FOCs yields

$$\frac{(1+\lambda) - \lambda H_i}{(1+\lambda) - \lambda H_l} \cdot \frac{u_{c_i}}{v_l} = \frac{F_i}{F_l}$$

where $H_i = -\frac{u_{c_i c_i} c_i}{u_{c_i}}$ and $H_l = -\frac{v_{ll} l}{v_l}$ are the elasticity of u_{c_i} and v_l

Implication for Taxes

- We know that
$$1+\tau_i^* = \frac{u_{c_i}}{v_l}\frac{F_l}{F_i}$$
- Therefore,
$$1+\tau_i^* = \frac{(1+\lambda)-\lambda H_l}{(1+\lambda)-\lambda H_i}$$
and after manipulating gives
$$\frac{\tau_i^*}{1+\tau_i^*} = \frac{\lambda(H_i - H_l)}{(1+\lambda)-\lambda H_l}$$
- Combining formulas for taxes on goods i and j gives:
$$\frac{\tau_i^*/(1+\tau_i^*)}{\tau_j^*/(1+\tau_j^*)} = \frac{H_i - H_l}{H_j - H_l}$$
So if $H_i > H_j > H_l$, then $\tau_i^* > \tau_j^*$

Implication for Taxes

- Suppose U is quasi-linear (no income effects):

$$U(c_1, ..., c_n, l) = \sum_{i=1}^{n} u_i(c_i) - \alpha l$$

  where $\alpha > 0$

- Then, $H_l = 0$ and

$$\frac{\tau_i^*/(1+\tau_i^*)}{\tau_j^*/(1+\tau_j^*)} = \frac{\varepsilon_i}{\varepsilon_j}$$

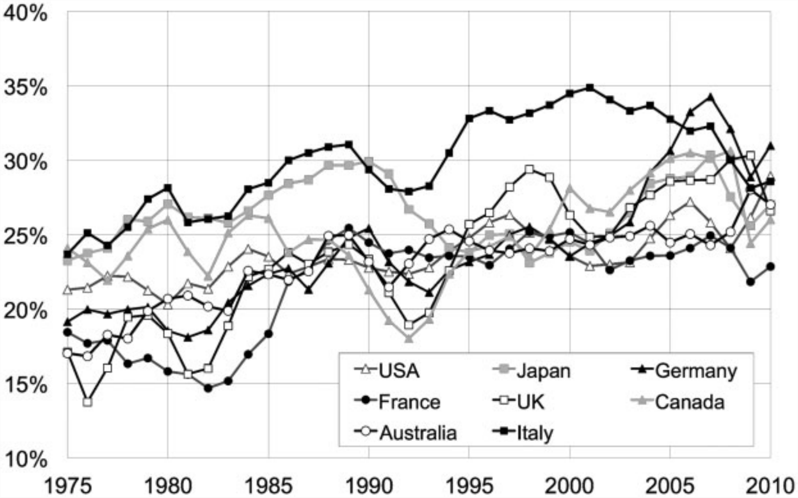
  where ε_i and ε_j are the elasticity of marginal utility and also the reciprocal of the price elasticity of demand!

- **Result**: tax more heavily the goods for which demand is inelastic

Dynamic Problem: Optimal Income Taxation

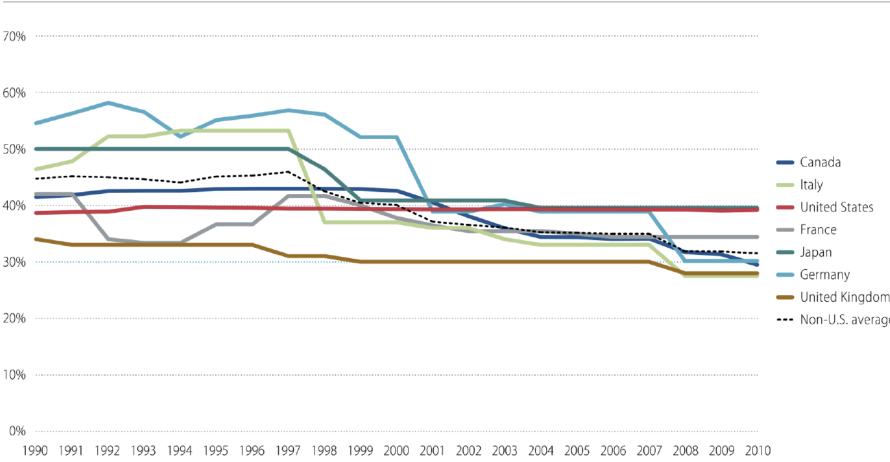
Without Uncertainty

Optimal Capital Taxation: Motivation



Capital-share in factor price national income. Piketty, Zucman, 2014

Optimal Capital Taxation: Motivation



Corporate Tax Rate. Auerbach, 2010.

Capital Taxes in Theory

- Most influential: Chamley (1986) and Judd (1985)'s **zero capital tax** result
  - In the **long-run**, the optimal linear capital income tax should be zero
  - perhaps even reflected in observed policy (see previous slide)
- This remarkable result asserts that capital income taxation serves neither **efficiency** nor **redistributive** purposes in the long run

Environment

- The preferences of the representative household are given by

$$\sum_{t=0}^{\infty}\beta^t u\left(c_t, l_t\right) \quad \beta \in (0,1) \tag{4}$$

where c_t and l_t are the consumption and leisure

- The two resource constraints are given by

$$\begin{aligned} l_t + n_t &= 1 \\ g_t + c_t + k_{t+1} &= F\left(k_t, n_t\right) + (1-\delta)\, k_t \end{aligned} \tag{5}$$

where $\{g_t\}_{t=0}^{\infty}$ is an exogenous sequence of government purchases

- Linear homogeneity of F implies

$$F\left(k_t, n_t\right) = F_k k_t + F_n n_t$$

where n_t and k_t are labor and capital

Warm up: First Best

- Benevolent Planner chooses allocations without restrictions other than technology. Formally,

$$V\left(\hat{k}_0\right) = \max_{\{c_t, l_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u\left(c_t, l_t\right) \quad \text{s.t.}$$

$$g_t + c_t + k_{t+1} = F\left(k_t, 1 - l_t\right) + \left(1 - \delta\right) k_t \quad \forall t \geq 0$$

$$c_t \geq 0, \quad k_t \geq 0 \quad \forall t \geq 0$$

$$k_0 = \hat{k}_0 \text{ given}$$

- With $\beta^t \lambda_t$ as the Lagrangian multiplier on the time t constraint, the FOCs are

$$c_t \quad : \quad u_{c,t} = \lambda_t \tag{6}$$

$$l_t \quad : \quad u_{l,t} = \lambda_t F_n \tag{7}$$

$$k_{t+1} \quad : \quad \lambda_t = \beta \lambda_{t+1} \left(F_k + 1 - \delta\right) \tag{8}$$

Warm up: First Best

- From (6) and (7), the efficiency **intratemporal** conditions is

$$u_{l,t} = u_{c,t} F_n \tag{9}$$

- From (6) and (8), the efficiency **intertemporal** conditions is

$$u_{c,t} = \beta u_{c,t+1} \left(F_k + 1 - \delta \right) \tag{10}$$

- From time-zero perspective

$$\beta^t \frac{u_{c,t}}{u_{c,0}} = \prod_{s=0}^{t-1} \frac{1}{F_k \left(k_{s+1}, n_{s+1} \right) + 1 - \delta} \equiv p_t$$

- To implement FB allocation, need lump sum taxes

Admissibility Constraints

- Main idea: Society faces more then just **feasibility** constraints in the allocation of resources
  - Feasibility constraints: Resource and technological constraints
- Possible additional constraints:
  - Ad hoc constraints on fiscal instruments
  - Lack of commitment (enforcement)
  - Private information
  - Political economy constraints
- Call these **admissibility** constraints

Second Best Allocation

- **First Best**: Optimal allocation under feasibility constraints
  - No distortions: MRS in consumption equal to the corresponding MRT
- **Second Best**: Optimal allocation under additional admissibility constraints. Variety of distortions arise:
  1. **Intratemporal** distortions
  $$\frac{u_{l,t}}{u_{c,t}} \neq F_n$$
  2. **Intertemporal** distortions:
  $$\frac{u_{c,t}}{\beta u_{c,t+1}} \neq F_k + 1 - \delta$$

Second Best Allocation

- Explore **static** and **intertemporal** distortions in second best economies
  - Static distortions: **Labor income taxes**
  - Intertemporal distortions: **Capital income taxes, variable consumption tax**
- Focus on intertemporal distortions:
  - When are intertemporal distortions ruled out in the second best?
  - What is different about economies in which they are optimal?
- Why care about intertemporal distortions?
  - Implications for capital accumulation, risk sharing, consumption inequality ...

Government

- The **government** is the new agent of the economy (together with a representative household and firm)
- It finances its stream of purchases $\{g_t\}_{t=0}^{\infty}$ by trading one-period bonds b_t and levying flat-rate, time-varying taxes on earnings from capital at rate τ_t^k and from labor at rate τ_t^n
- The Gvt's budget constraint is

$$g_t + b_t = \tau_t^n w_t n_t + \tau_t^k r_t k_t + \frac{b_{t+1}}{R_t} \tag{11}$$

where R_t is the gross rate of return on one-period bonds from t to $t+1$

Households

- The household can save in Gvt debt b_t and in capital k_t
- It maximizes its utility (4) s.t. the following sequence of budget constraints

$$c_t + k_{t+1} + \frac{b_{t+1}}{R_t} = (1 - \tau_t^n) w_t n_t + (1 - \tau_t^k) r_t k_t + (1 - \delta) k_t + b_t \tag{12}$$

- With $\beta^t \lambda_t$ as the Lagrangian multiplier on the time t budget constraint, the FOCs are

$$c_t \quad : \quad u_{c,t} = \lambda_t \tag{13}$$

$$l_t \quad : \quad u_{l,t} = \lambda_t (1 - \tau_t^n) w_t \tag{14}$$

$$k_{t+1} \quad : \quad \lambda_t = \beta \lambda_{t+1} [(1 - \tau_{t+1}^k) r_{t+1} + 1 - \delta] \tag{15}$$

$$b_{t+1} \quad : \quad \frac{\lambda_t}{R_t} = \beta \lambda_{t+1} \tag{16}$$

Households' Equilibrium Condition I

- Substituting (13) into (14) and (15) yields

$$u_{l,t} = (1 - \tau_t^n) w_t u_{c,t} \tag{17}$$

and

$$u_{c,t} = \beta u_{c,t+1}[(1 - \tau_{t+1}^k) r_{t+1} + 1 - \delta] \tag{18}$$

- We say that labor and capital taxes are distortionary. To see that, compare the efficient conditions (9) and (10) to the competitive equilibrium conditions (17) and (18)

  $\Rightarrow$ the allocation is **efficient** only if $\tau_t^n = \tau_t^k = 0 \ \forall t \geq 0$.

Households' Equilibrium Condition II

- Moreover, (15) and (16) implies

$$R_t = (1 - \tau^k_{t+1}) r_{t+1} + 1 - \delta \tag{19}$$

- Because only one financial asset is needed to accomplish all intertemporal trades in a world without uncertainty, (19) is a **no-arbitrage** condition for trades in capital and bonds that ensures that the two assets have the same rate of return
  - $\Rightarrow$ Ensure the existence of a Competitive Equilibrium with bounded budget sets

HH's Present Value Budget Constraint

- Consolidating two consecutive budget constraints:

$$c_t + \frac{c_{t+1}}{R_t} + \frac{k_{t+2}}{R_t} + \frac{b_{t+2}}{R_t R_{t+1}} = (1-\tau_t^n) w_t n_t$$

$$+ \frac{(1-\tau_{t+1}^n) w_{t+1} n_{t+1}}{R_t} + \Big[\frac{(1-\tau_{t+1}^k) r_{t+1} + 1 - \delta}{R_t} - 1\Big] k_{t+1}$$

$$(1-\tau_t^k) r_t k_t + (1-\delta) k_t + b_t$$

- Continuing the process yields:

$$\sum_{t=0}^{\infty} q_t c_t = \sum_{t=0}^{\infty} q_t (1-\tau_t^n) w_t n_t + \left[\left(1-\tau_0^k\right) r_0 + 1 - \delta\right] k_0 + b_0$$

where $q_t = \prod_{s=0}^{t-1} R_s^{-1}$ with $q_0 = 1$

Firms

- In each period, the representative firm takes (r_t, w_t) as given, rents capital and labor from households, and maximizes profits,

$$\Pi = F(k_t, n_t) - r_t k_t - w_t n_t$$

- The FOCs for this problem are

$$F_k(k_t, n_t) = r_t \tag{20}$$

$$F_n(k_t, n_t) = w_t \tag{21}$$

Definitions

Definition: A **feasible allocation** is a sequence $(k_t, c_t, n_t, g_t)_{t=0}^{\infty}$ such that the resource constraint (5) holds $\forall t \geq 0$

Definition: A **price system** is a non-negative bounded sequence $(w_t, r_t, R_t)_{t=0}^{\infty}$

Definition: A **government policy** system is a sequence $\left(g_t, \tau_t^k, \tau_t^n, b_t\right)_{t=0}^{\infty}$

... and we can finally define a competitive equilibrium.

Competitive Equilibrium

Definition: A **competitive equilibrium** is a feasible allocation, a price system, and a Gvt policy such that

i. given the price system and the Gvt policy, the allocation solves both the firm's problem and the HH's problem;

ii. given the allocation and the price system, the Gvt policy satisfies the sequence of Gvt budget constraints (11)

There are **many** competitive equilibria, indexed by different Gvt policies. This multiplicity motivates the Ramsey problem

Definition: Given k_0 and b_0, the **Ramsey problem** is to choose a competitive equilibrium that maximizes expression (4)

Observations

Observation 1: Tax Equivalence $\Rightarrow$ With commodity tax, Focs:

$$u_{l,t} = \frac{(1-\tau_t^n)}{(1+\tau_t^c)} w_t u_{c,t}$$

and

$$\frac{u_{c,t}}{(1+\tau_t^c)} = \beta \frac{(u_{c,t+1})}{(1+\tau_{t+1}^c)} (1-\tau_{t+1}^k) r_{t+1}$$

- These tax instruments are not all independent!

  $\Rightarrow$ Without loss of generality, we set $\tau_t^c = 0$ for all t

- Alternatively, we could set $\tau_t^k = 0$ for all t and have

$$\frac{1+\tilde{\tau}_t^c}{1+\tilde{\tau}_{t+1}^c} = 1-\tau^k \Rightarrow 1+\tilde{\tau}_t^c < 1+\tilde{\tau}_{t+1}^c$$

- Positive tax on capital is equivalent to an increasing tax on consumption

Observations

Observation 2: Non-distortionary taxation of capital in period 0

- Taxes on capital in period 0 does not distort any decisions
  - $\Rightarrow$ Equivalent to a lump sum tax
- If government could use this tax, it would set it at a very high level to get enough revenues to finance all future g_t
- Assume (to be justified later on) that this tax is unavailable to make the problem interesting

$$\tau_0^k = 0$$

Observations

Observation 3: Nothing fancy about dynamics

- Instead of thinking about period t consumption, think about period 0 consumption of a good with label "t"
  - $\Rightarrow$ Equivalent to the static commodity taxation problem with infinitely many goods

Observation 4: Many ways to ensure that distortions hold (see Observation 1)

- Here: tax gross return on capital $1 + r - \delta$
  - $\Rightarrow$ Could instead tax net return $r - \delta$ (as is usually done in practice): nothing changes in the analysis

Ramsey Approach to Optimal Policies

1. Benevolent government
2. Chooses policies once and for all under **full commitment**
3. Has access to proportional labor and capital income taxes, can borrow and lend
4. Exogenous government consumption or outstanding debt $\Rightarrow$ Tax revenues must be collected
5. Takes as given private sector behavior in response to different policies

Chamley (1986)

- Solving a Ramsey plan is often about **rewriting it.** We do so here, using **after-tax prices** (rental rate of capital and wage rate) rather than pre-tax prices

$$\begin{aligned} \tilde{r}_t &\equiv (1-\tau_t^k) r_t \\ \tilde{w}_t &\equiv (1-\tau_t^n) w_t \end{aligned}$$

  and using firm's conditions (20) and (21)

$$\begin{aligned} \tau_t^n w_t n_t + \tau_t^k r_t k_t &= (r_t - \tilde{r}_t) k_t + (w_t - \tilde{w}_t) n_t \\ &= F_k k_t + F_n n_t - \tilde{r}_t k_t - \tilde{w}_t n_t \\ &= F(k_t, n_t) - \tilde{r}_t k_t - \tilde{w}_t n_t \end{aligned}$$

- This allows to get rid of the choice for τ_t^k and τ_t^n when solving
  - we will solve for the taxes in a second step, after finding the rest of the control variables

Lagrangian

The Ramsey problem in Lagrangian form becomes

$$
\begin{aligned}
\mathcal{L} \;=\; \sum_{t=0}^{\infty}\beta^t\{ & u\left(c_t, 1-n_t\right) \qquad (22)\\
& +\psi_t[F\left(k_t, n_t\right) - \tilde{r}_t k_t - \tilde{w}_t n_t + \frac{b_{t+1}}{R_t} - g_t - b_t]\\
& +\theta_t[F\left(k_t, n_t\right) + \left(1-\delta\right) k_t - g_t - c_t - k_{t+1}]\\
& +\mu_{1t}[u_{l,t} - u_{c,t}\tilde{w}_t]\\
& +\mu_{2t}[u_{c,t} - \beta u_{c,t+1}(\tilde{r}_{t+1} + 1 - \delta)]\}
\end{aligned}
$$

where $R_t = \tilde{r}_t + 1 - \delta$

Remark: (12) is not explicitly included because it is redundant when the Gvt satisfies (11) and (5) holds

Zero Capital Tax

The FOC with respect to k_{t+1} is

$$\theta_t = \beta\{\psi_{t+1}[F_{k_{t+1}} - \tilde{r}] + \theta_{t+1}[F_{k_{t+1}} + 1 - \delta]\} \tag{23}$$

The equation has a straightforward interpretation:

- A marginal increment of capital investment at t increases the quantity of available goods at $t+1$ by $[F_{k_{t+1}} + 1 - \delta]$, whose social marginal value is θ_{t+1}
- In addition, the increase in tax revenues is $[F_{k_{t+1}} - \tilde{r}]$, which enables the Gvt to reduce debt or other taxes by the same amount. The reduction of excess **burden** is $\psi_{t+1}[F_{k_{t+1}} - \tilde{r}]$
- The sum of the two effects in $t+1$ is discounted by β and set equal to the social marginal value of the initial investment good in t, which is given by θ_t

Zero Capital Tax

- Suppose that the Ramsey problem converges to a steady state (i.e., $c_t \to c$, $k_t \to k$ and $l_t \to l$)
- Using (20), the steady-state version of (23) is

$$\theta = \beta\{\psi[r - \tilde{r}] + \theta[r + 1 - \delta]\}$$

- Now the HH's optimality condition for the choice of capital in (15) is

$$1 = \beta(\tilde{r} + 1 - \delta)$$

- Combining this two yields $(\psi + \theta)(r - \tilde{r}) = 0$
- Given that budget constraints (5) and (11) bind under very general conditions, this implies $r = \tilde{r}$ so that $\tau^k = 0$

Zero Capital Tax

Proposition: If there exists a steady-state Ramsey allocation, the associated limiting tax rate on capital is zero

$\Rightarrow$ Tax labor and/or consumption, but not capital!

Remark: This conclusion is robust to whether the government can issue debt or must run a balanced budget in each period (Hint: set $b_t = b_{t+1} = 0$ in (22))

Fact 1: It is important to keep in mind that a limiting zero capital tax pertains only if there is a steady state (Straub and Werning, 2019)

Fact 2: The analysis is silent about how much redistribution is accomplished in the transition period

Zero Capital Tax: Intuition

- Taxing capital income in period $t+1$ is equivalent to taxing consumption at a higher rate in $t+1$ than in t
- If a steady state exists, intertemporal elasticity is constant in the long run
- From the optimal uniform commodity taxation thrm, it is not optimal to differentially tax consumption over time
- If utility separable additive and constant elasticity of substitution, $\tau_t^k = 0$ for any $t \geq 1$

Relevance and Significance

- No Permanent Intertemporal Distortions
- Very robust result in Ramsey models with
  - Heterogeneous agents: Judd 1987
  - Aggregate shocks: Zhu 1992
  - Human capital accumulation: Jones, Manuelli, and Rossi 1997
  - Incomplete markets: Farhi 2006
  - See Atkeson, Chari, Kehoe (1999) for additional examples
- Intuition?
  - Second best theory typically requires distortions to be spread along all margins
  - To provide intuition, we will derive the result with an alternative argument

Primal Approach

- In (22), a pair of taxes (τ_t^k, τ_t^n) and prices (r_t, w_t) is reduced to a pair of numbers $(\tilde{r}_t, \tilde{w}_t)$ using the firm's FOCs and equilibrium outcomes in factor markets
- We now get rid of all prices and taxes so that the Gvt can be directly choose a feasible allocation, s.t.
  - constraints ensuring the existence of prices and taxes consistent with the optimization behavior of HHs and firms
- This new problem is termed the **Primal Approach** since the Gvt directly chooses quantities (and τ_0^k), instead of choosing tax rates and thereby indirectly influencing quantities

Stepwise Summary of the Primal Approach

1. Obtain the HH's and the firm's FOCs, as well as any arbitrage conditions. Solve these conditions for $\{q_t, r_t, w_t, \tau_t^k, \tau_t^n\}_{t=0}^{\infty}$ as functions of allocation $\{c_t, n_t, k_{t+1}\}_{t=0}^{\infty}$
2. Substitute these expressions for taxes and prices in terms of the allocation into the HH's present-value budget constraint. This is an intertemporal constraint involving only allocation
3. Solve for the Ramsey allocation by maximizing (4) subject to (5) and the **implementability condition** derived in step 2
4. After the Ramsey allocation is solved, use the formulas from step 1 to find taxes and prices

Step 1

- Let λ be a Lagrange multiplier on HH budget constraint:

$$\begin{aligned} c_t &: \quad \beta^t u_{c,t} - \lambda q_t = 0 \\ l_t &: \quad -\beta^t u_{l,t} + \lambda q_t (1 - \tau_t^n) w_t = 0 \end{aligned}$$

  implying

$$q_t = \beta^t \frac{u_{c,t}}{u_{c,0}}, \quad \text{(from time-0 perspective)} \tag{24}$$

$$(1 - \tau_t^n) w_t = \frac{u_{l,t}}{u_{c,t}} \tag{25}$$

- The no arbitrage condition reads

$$\frac{q_t}{q_{t+1}} = \left(1 - \tau_{t+1}^k\right) r_{t+1} + 1 - \delta \tag{26}$$

- Profit maximization and factor market equilibrium imply (20) and (21)

Steps 2 & 3

- Step 2: Substituting (24) and (25) and $r_0 = F_{k,0}$ yields the **implementability constraint**

$$\sum_{t=0}^{\infty}\beta^t[u_{c,t}c_t - u_{l,t}n_t] = u_{c,0}\{[(1-\tau_0^k)F_{k,0}+1-\delta]k_0+b_0\} \quad (27)$$

- Step 3: The Ramsey problem is to

$$\begin{aligned} & \max_{\{c_t,n_t,k_{t+1}\}_{t=0}^{\infty}}\sum_{t=0}^{\infty}\beta^t u(c_t,l_t) \quad \text{s.t.} \\ & \text{Eqs. } (5),\ (27),\ \text{and} \\ c_t &\geq 0,\ n_t \geq 0,\ k_{t+1} \geq 0, \end{aligned}$$

with given k_0, b_0 and τ_0^k

Ramsey Allocation Problem

- Let Φ be a Lagrangian multiplier and define

$$V(c_t, l_t, \Phi) = u(c_t, l_t) + \Phi\left[u_{c,t}c_t - u_{l,t}n_t\right]$$

  the **modified** objective fct incorporating constraints on allocation from proportional taxes

- Government solves

$$\max_{\{c_t, n_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left\{ V(c_t, l_t, \Phi) + \theta_t \begin{bmatrix} F(k_t, n_t) + (1-\delta)k_t \\ -g_t - c_t - k_{t+1} \end{bmatrix} \right\} \tag{28}$$
$$-\Phi u_{c,0}\left\{\left[\left(1-\tau_0^k\right)F_{k,0} + 1 - \delta\right]k_0 + b_0\right\}$$

  - $\{\theta_t\}_{t=0}^{\infty} \geq 0$: multiplier on resource constraint
  - $\Phi \geq 0$: multiplier on implementability constraint

Necessary Conditions

Optimality conditions for $t > 0$

$$\begin{aligned} c_t &: \quad V_{c,t} = \theta_t \\ n_t &: \quad V_{n,t} = -\theta_t F_{n_t} \\ k_{t+1} &: \quad \theta_t = \beta\theta_{t+1}(F_{k_{t+1}} + 1 - \delta) \end{aligned}$$

Intratemporal wedge

$$-\frac{V_{n,t}}{V_{c,t}} = F_{n_t} \Rightarrow -\frac{u_{n,t}}{u_{c,t}} \neq F_{n_t} \text{ if } \Phi > 0 \tag{29}$$

Intertemporal wedge

$$\frac{V_{c,t}}{\beta V_{c,t+1}} = F_{k_{t+1}} + 1 - \delta \Rightarrow \frac{u_{c,t}}{\beta u_{c,t+1}} \neq F_{k_t} + 1 - \delta \text{ if } \Phi > 0 \tag{30}$$

Necessary Conditions

For $t = 0$ the FOCs are different (reflecting the time inconsistency of the choice)

$$\begin{aligned} c_0 &: \quad V_{c,0} = \theta_0 + \Phi A_c \\ n_0 &: \quad V_{n,0} = -\theta_0 F_{n_0} + \Phi A_n \end{aligned}$$

where $A \equiv u_{c,0} \left\{ \left[\left(1 - \tau_0^k\right) F_{k,0} + 1 - \delta \right] k_0 + b_0 \right\}$. Rearranging

Intratemporal wedge

$$V_{n,0} = [\Phi A_c - V_{c,0}] F_{n_0} + \Phi A_n \tag{31}$$

Intertemporal wedge

$$V_{c,0} - \Phi A_c = \beta V_{c,1} (F_{k_1} + 1 - \delta) \tag{32}$$

Step 4

After an allocation $(c_t, n_t, k_{t+1})_{t=0}^{\infty}$ has been found using (29)-(32) together with (5) and (27), obtain

- q_t from (24):
$$q_t = \beta^t \frac{u_{c,t}}{u_{c,0}},$$
- r_t and w_t from $F_k(k_t, n_t) = r_t$ and $F_n(k_t, n_t) = w_t$
- τ_t^n from combining $(1-\tau_t^n)F_{n_t} = \frac{u_{l,t}}{u_{c,t}}$ and (29) so that
$$\tau_t^n = 1 - \frac{u_{l,t}}{V_{n,t}} \frac{V_{c,t}}{u_{c,t}}$$
- and finally τ_t^k from $\frac{q_t}{q_{t+1}} = \left(1-\tau_t^k\right) F_{k_t} + 1 - \delta$

Revisiting a Zero Capital Tax

- Assume: (i) there is a $T \geq 0$ for which $g_t = g$ for all $t \geq T$ and (ii) there exists a solution to the Ramsey problem and that it converges to a time-invariant allocation
- Then, the stationary version of (30) implies

$$1 = \beta(F_k + 1 - \delta) \tag{33}$$

- Since c is constant in the limit, (24) implies that $\frac{q_t}{q_{t+1}} \to \beta^{-1}$ as $t \to \infty$. Then the no-arbitrage condition for capital (26) is

$$1 = \beta[(1 - \tau_t^k)F_k + 1 - \delta] \tag{34}$$

- (33) and (34) imply that $\tau^k = 0$

Taxation of Initial Capital

- We have set $\tau_0^k = 0$ (or some other small fixed number). Now suppose that the Gvt is free to choose it
- The FOC of the Gvt problem (28) with respect to τ_0^k is

$$\Phi u_{c,0} F_{k,0} k_0 = 0 \tag{35}$$

which is strictly positive $\forall \tau_0^k$ as long as $\Phi > 0$, a measure of the utility costs of raising Gvt revenues via distorting taxes

- Without distortionary taxation, a competitive equilibrium attains the first-best and $\Phi = 0$, so that the budget constraint does not exert any additional constraining effect on welfare maximization beyond what is present in the economy's technology

Taxation of Initial Capital

- In contrast, when the Gvt has to use $\left\{\tau_t^n, \tau_{t+1}^k\right\}$, Φ and reflects the welfare cost of the distorted margins
- By raising τ_0^k and increasing the revenues from lump-sum taxation of k_0, the Gvt reduces its need to rely on future distortionary taxation, and hence Φ falls
- It follows from (35) that the Gvt should set τ_0^k high enough to drive Φ down to 0
  - The scheme may involve $\tau_0^k > 1$ for high values of g_t and b_0
- In other words, the Gvt should raise all revenues through a time-0 capital levy, then lend the proceeds to the private and finance Gvt expenditures via the interest from the loan
  - Gvt sets $\tau_t^n = 0$ for all $t \geq 0$ and $\tau_t^k = 0$ for all $t \geq 1$

Limited Tax Instruments

- Suppose that production (with CRS) uses an additional factor in fixed supply $z_t = Z$ which cannot be taxed $\tau_z = 0$:

$$F(k_t, n_t, z_t) \quad \text{with } p_t^z = F_{z_t}$$

- How does the Ramsey problem change? The main difference is in the present value of individual budget constraint with an additional stream of revenue

$$\sum_{t=0}^{\infty} \beta^t [u_{c,t}[c_t - F_{z,t} Z] - u_{l,t} n_t] = u_{c,0}\{[(1-\tau_0^k)F_{k,0} + 1 - \delta]k_0 + b_0\}$$

- Assuming the existence of a steady state: Capital taxes will be different from zero in the long run. Why?

Limiting Economy (Straub and Werning, 2015)

- Chamley-Judd models of optimal taxation assume that a steady state exists
- In particular they assume that endogenous multipliers (on resource constraints) are bounded and converge $\theta_t \to \theta$. From Eq. (30):

$$\frac{\theta_t}{\beta\theta_{t+1}} = F_{k_{t+1}} + 1 - \delta$$

- Straub and Werning (2015) show that if $IES < 1$ and Φ sufficiently large, then θ_t diverges to infinity and τ_t^k cannot go to zero
- Intuition: When $IES < 1$ (income effect dominates substitution effect), announced tax increase raises saving today

Time Consistency

- Compute the optimal policy from $t = 0$ perspective
  - $\Rightarrow$ high capital taxes early, go to zero (say, by $t = 100$)
- Compute the optimal policy from $t = 100$ perspective
- The two will not be the same
  - $\Rightarrow$ Government has strong incentives to deviate from the optimal policy and choose different taxes later on

In later periods, government always tempted to tax capital at high level ...

... having promised in earlier periods that it will not!

Time Consistency

- If government cannot commit, agents will take it into account when taxes are announced
- Invest less, even though they are promised low taxes tomorrow
  - $\Rightarrow$ Know that tomorrow government cannot keep its promise and will revert to high taxes
- Welfare losses can be large without commitment Kydland and Prescott's 2004 Nobel prize
- Lesson: Optimal Ramsey taxation is not time consistent. It is important to be able to commit and avoid temptation ex-post
- For Ramsey taxation without commitment see Benhabib and Rustichini (1997), Klein and Rios-Rull (2008)

Dynamic Problems Under Uncertainty

A Stochastic Economy

Now, add **shocks** to our economy (populated by a representative household, a representative firm and a Gvt)

- The Gvt must finance a stochastic stream of consumption expenditures $\{g(s_t)\}_{t\geq 0}$, where s_t denotes the realization of an exogenous state in the history $s^t = [s_0, s_1, ..., s_t]$
  - Gvt consumption is the only source of uncertainty with $\pi(s^t)$ being the (unconditional) probability of history s^t occurring (no Markov structure)
- Assumptions:
  - **No capital**: the Gvt can only use labor taxes and public debt
  - **Complete markets**: the Gvt and the households issue state-contingent bonds

Environment

- The representative HH's preferences are ordered by

$$\sum_{t=0}^{\infty} \beta^t \sum_{s^t} \pi(s^t) u\left(c(s^t), l(s^t)\right) \tag{36}$$

  where $c(s^t)$ and $l(s^t)$ are the consumption and leisure at time t give history s^t

- Firm produces the consumption good with linear technology

$$y(s^t) \leq n(s^t)$$

- The two resource (feasibility) constraints are

$$\begin{aligned} l(s^t) + n(s^t) &= 1 \\ g(s_t) + c(s^t) &= 1 - l(s^t) \end{aligned} \tag{37}$$

- A **feasible** allocation is a plan $\{g(s_t), c(s^t), l(s^t)\}_{t \geq 0}$ such that (37) holds $\forall t \geq 0$ and $\forall s^t$

Warm up: First Best

- The planner's object is

$$\max_{\{c(s^t),l(s^t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \pi(s^t) u[\big(c(s^t), l(s^t)\big)] \quad \text{s.t.}$$

$$g(s_t) + c(s^t) + l(s^t) = 1 \quad \forall t \geq 0 \text{ and } \forall s^t$$

- The Lagrangian of this problem is

$$\sum_{t=0}^{\infty} \beta^t \sum_{s^t} \pi(s^t) \{u[\big(c(s^t), l(s^t)\big)] + \lambda(s^t)[1 - g(s_t) - c(s^t) - l(s^t)]\}$$

- The associated FOCs are

$$c_t \quad : \quad u_{c,t} = \lambda(s^t) \tag{38}$$

$$l_t \quad : \quad u_{l,t} = \lambda(s^t) \tag{39}$$

Warm up: First Best

- From (38) and (39), the efficiency conditions is

$$u_c(c(s^t), 1 - g(s_t) - c(s^t)) = u_l(c(s^t), 1 - g(s_t) - c(s^t))$$

- $c(s^t)$ is fully pinned down by the above condition and it only depends on $g(s_t)$ at that node, i.e., the most recent element of the history s_t
    - Hence, the first-best allocation is **history independent**
- We now enter into Lucas and Stokey (1983):
    i. Gvt can only use distortionary tools,
    ii. Gvt chooses the best competitive equilibrium

Government

- Given s^t, the Gvt finances its exogenous purchase and debt obligations by levying flat-rate taxes on labor earnings at rate $\tau^n(s^t)$ and by issuing state-contingent debt
- Let $b(s_{t+1} \mid s^t)$ be Gvt indebtedness at the beginning of $t+1$
  - This state-contingent asset is traded in t at the price $p(s_{t+1} \mid s^t)$, in terms of time-t goods
- The Gvt's budget constraint is

$$g(s_t) + b(s_t \mid s^{t-1}) = \tau^n(s^t)w(s^t)n(s^t) + \sum_{s_{t+1}} p(s_{t+1} \mid s^t)b(s_{t+1} \mid s^t)$$

where $w(s^t)$ is the wage and b_0 is given

- With time-0 trading, the budget constraint is

$$\sum_{t=0}^{\infty}\sum_{s^t} q(s^t)[g(s_t) - \tau^n(s^t)w(s^t)n(s^t)] = -b_0 \qquad (40)$$

Households

- The HH maximizes its utility (36) s.t. the sequence of budget constraints

$$\begin{aligned} c(s^t)+\sum_{s_{t+1}} p(s_{t+1} \quad | \quad & s^t) b(s_{t+1} | s^t) \\ \leq \quad & (1-\tau^n(s^t)) w(s^t) n(s^t) + b(s_t | s^{t-1}) \end{aligned}$$

with $b(s_0)$ is given

- In competitive equilibrium, the debt of the Gvt equals the assets of the household. With time-0 trading, the budget constraint is

$$\sum_{t=0}^{\infty} \sum_{s^t} q(s^t)[c(s^t) - (1-\tau^n(s^t)) w(s^t) n(s^t)] = b_0 \qquad (41)$$

Individual Optimal Allocation

- The FOCs for the HH's problem imply

$$\frac{u_l(s^t)}{u_c(s^t)} = (1-\tau^n(s^t))w(s^t) \quad (42)$$

$$q(s^t) = \beta^t \pi(s^t)\frac{u_c(s^t)}{u_c(s_0)} \quad \text{(from } t=0 \text{ perspective)} \quad (43)$$

- The firm optimization implies that labor factor is paid its marginal product

$$w(s^t) = 1 \quad (44)$$

- Note there are many competitive equilibrium allocations which can be indexed by different government policies

Ramsey Allocation Problem

- Now ask what competitive allocation should be chosen by a benevolent Gvt (Ramsey problem for the stochastic economy)
  - The computational strategy is the same as for a deterministic economy

Step 2: To get rid of taxes and prices, substitute (42) and (43) into (41) and obtain the **implementability** constraint

$$\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi(s^t)[u_c(s^t)c(s^t)-u_l(s^t)n(s^t)]=u_c(s_0)b_0 \quad (45)$$

Step 3: The Ramsey allocation problem is to maximize (36) by choice of $\{c(s^t), l(s^t)\}$, subject to (37) and (45). Then form the Lagrangian in the same way as before

Lagrangian for the Ramsey Problem

- Let Φ and $\theta(s^t)$ be the multipliers on (37) and (45) at node s^t and the modified object fct $V(c(s^t), l(s^t), \Phi)$ be equal to

$$u[(c(s^t), l(s^t))] + \Phi\left[u_c(s^t)c(s^t) - u_l(s^t)n(s^t)\right]$$

- The Lagrangian for the Ramsey allocation problem is:

$$\sum_{t=0}^{\infty}\beta^t\sum_{s^t}\pi(s^t)\{V(c(s^t), l(s^t), \Phi)$$

$$+\theta(s^t)[g(s_t) + c(s^t) + l(s^t) - 1]\} - \Phi u_c(s_0)b_0$$

Necessary Conditions

$$c(s^t): V_c(s^t) - \theta(s^t) = 0, \quad t \geq 1$$

$$n(s^t): V_n(s^t) + \theta(s^t) = 0, \quad t \geq 1$$

$$c(s_0): V_c(s_0) - \theta(s_0) - \Phi u_{cc}(s_0) b_0 = 0, \quad t = 0$$

$$n(s_0): V_n(s_0) + \theta(s_0) - \Phi u_{cl}(s_0) b_0 = 0, \quad t = 0$$

where

$$\begin{aligned} V_c(\cdot) &= (1+\Phi) u_c(\cdot) + \Phi[u_{cc}(\cdot)c(\cdot) - u_{lc}(\cdot)n(\cdot)] \\ V_n(\cdot) &= -(1+\Phi) u_l(\cdot) - \Phi[u_{cl}(\cdot)c(\cdot) - u_{ll}(\cdot)n(\cdot)] \end{aligned}$$

Note that the fact that FOCs in $t = 0$ are different from those in $t \geq 1$ implies that there is time inconsistency [it can be shown with the recursive form]

History Independence

- Not possible to prove in general whether the FOCs are also sufficient or whether a solution exists
  - If the optimum of the Ramsey allocation problem exists and is interior, it will satisfy this set of restrictions
- Note that for $t > 0$:

$$V_c(s^t) = -V_n(s^t)$$

At each node, the optimal allocation only depends on $g(s_t)$, i.e., on s_t only

  - To see this, replace $l(s^t)$ with (37) which leaves one unknown, $c(s^t)$, as a function of $g(s_t)$

History Independence

- To see the properties of optimal amount of state-contingent debt, consider the Gvt budget constraint:

$$b(s_t|s^{t-1}) = \sum_{j=0}^{\infty} \sum_{s^{t+j}|s^t} \beta^j \pi(s^{t+j}|s^t) \frac{u_c(s^{t+j})}{u_c(s^t)} \Big\{ \Big[1 - \frac{u_l(s^{t+j})}{u_c(s^{t+j})}\Big] \cdot [c(s^{t+j}) + g(s^{t+j})] - g(s^{t+j}) \Big\}$$

- Since $c(s^t)$ and $n(s^t)$ are a function of only current $g(s_t)$, if Markov process of shocks, then also $b(s_t|s^{t-1})$ is a function of only $g(s_t)$ [check RHS of budget is just a function of s_t]

Tax Smoothing

- Ramsey allocation is **history independent** for $t > 0$ thanks to the full-insurance provided by complete markets
- Tax distortions are the same in each period and debt is used to smooth tax distortion over time
- If $g_t = g$ for all t, the intra-temporal FOC is the same in each time $\Rightarrow (c_t, n_t) = (\bar{c}, \bar{n})$, $\tau_t^n = \bar{\tau}^n$ for each t and thus $b_t = 0$
- If $g_t = 0$ for all $t \neq T$ and $g_T > 0$ (non stochastic) $\Rightarrow$ $\tau_t^n = \bar{\tau}^n$ for each $t \neq T$ and constant tax revenue; surplus at any $t < T - 1$ to buy bonds; g_T financed by selling bonds and issuing new bonds rolled over whose interest covered by the constant tax revenue

Stochastic Economy with Capital

- With physical capital the sequence of HH budget constraints are:

$$c(s^t) + k_{t+1}(s_t) + \sum_{s_{t+1}} p(s_{t+1} \mid s^t) b(s_{t+1} \mid s^t)$$
$$\leq (1 - \tau^k(s^t)) r(s^t) k_t(s^{t-1}) + (1 - \delta) k_t(s^{t-1}) +$$
$$(1 - \tau^n(s^t)) w(s^t) n(s^t) + b(s_t \mid s^{t-1})$$

Stochastic Economy with Capital

- With physical capital the sequence of HH budget constraints are:

$$c(s^t) + k_{t+1}(s_t) + \sum_{s_{t+1}} p(s_{t+1} \mid s^t) b(s_{t+1} \mid s^t)$$
$$\leq (1 - \tau^k(s^t)) r(s^t) k_t(s^{t-1}) + (1 - \delta) k_t(s^{t-1}) +$$
$$(1 - \tau^n(s^t)) w(s^t) n(s^t) + b(s_t \mid s^{t-1})$$

- A competitive equilibrium $(c(s^t), n(s^t), k_{t+1}(s^t))_{t=0}^{\infty}$ can be implemented by a unique $\tau^n(s^t)$ but infinitely many plans for capital taxes and state-contingent debt: Verify it!
- A steady state of a stochastic economy is a stationary equilibrium, that is, an equilibrium where the process (s_t, k_t) is stationary ergodic Markov process on a compact set

Lesson from Stochastic Economy with Complete Market

- With or without physical capital, if s follows a Markov process, the end-of-period government debt is a function of the current state (s_t without capital or (s_t, k_t) with capital)
- Past history of shocks to government expenditure is not relevant
- It contradicts Barro (1979) intuition: government debt should be cointegrated with tax revenue and follow a random walk (Sargent and Velde, 1995, for evidence on US debt)
- Incomplete market models (Aiyagari et al., 2002) predict optimal policy more in line with Barro

Stochastic Economy with Incomplete Market

- Replace state-contingent bonds by risk-free bond $b_{t+1}(s^t)$ with risk-free gross interest rate $R(s^t)$

$$
\begin{aligned}
b_t(s^{t-1}) &= \tau^n(s^t)n(s^t) - g(s_t) - T(s^t) + \frac{b_{t+1}(s^t)}{R(s^t)} \\
&\equiv z(s^t) + \frac{b_{t+1}(s^t)}{R(s^t)}
\end{aligned}
$$

- Now it might be necessary to rebate through lump sum transfers because bonds are not state contingent
- We rule out Ponzi schemes by introducing natural debt limits

Stochastic Economy with Incomplete Market

- The HH FOCs are:

$$\frac{u_l(s^t)}{u_c(s^t)} = (1 - \tau^n(s^t))$$

$$\frac{1}{R(s^t)} = \sum_{s_{t+1}|s^t} p(s_{t+1}|s^t) = \sum_{s_{t+1}|s^t} \beta\pi(s_{t+1}|s^t)\frac{u_c(s^{t+1})}{u_c(s^t)}$$

- Replacing in the budget constraint, we get the implementability constraint:

$$u_c(s^t)c(s^t) + b_{t+1}(s_t)\sum_{s_{t+1}|s^t} \beta\pi(s_{t+1}|s^t)u_c(s^{t+1})$$

$$= u_c(s^t)b_t(s^{t-1}) + u_c(s^t)T(s^t) + u_l(s^t)n(s^t)$$

- Since markets are incomplete we have a sequence of IC for each s^t, t

- The Lagrangian of the Ramsey problem (considering interior solution) is

$$\sum_{t=0}^{\infty} \beta^t \sum_{s^t} \pi(s^t) \Big\{ u(c(s^t), l(s^t)) + \theta(s^t)[g(s_t) + c(s^t) + l(s^t) - 1] +$$

$$\Phi(s^t) \Big[u_c(s^t) c(s^t) + b_{t+1}(s_t) \sum_{s_{t+1}|s^t} \beta \pi(s_{t+1}|s^t) u_c(s^{t+1})$$

$$- (u_c(s^t) b_t(s^{t-1}) + u_c(s^t) T(s^t) + u_l(s^t) n(s^t)) \Big] \Big\}$$

Stochastic Economy with Incomplete Market

- The FOCs are:

$$c(s^t) : u_{c,t}(1+\Phi_t) + \Phi_t[u_{cc,t}c_t - u_{lc,t}n_t] + b_t u_{cc,t}[\Phi_{t-1} - \Phi_t] = \theta_t$$
$$l(s^t) : u_{l,t}(1+\Phi_t) + \Phi_t[u_{cl,t}c_t - u_{ll,t}n_t] + b_t u_{cl,t}[\Phi_{t-1} - \Phi_t] = \theta_t$$

- If Φ_t is no constant, then b_t influences $(c(s^t), l(s^t))$
- If Φ_t is constant, FOCs are as in the complete market case
- To know the path of Φ_t consider the FOC w.r.t. $b_{t+1}(s^t)$:

$$b_{t+1} : \Phi_t \mathbb{E}_t[u_{c,t+1}] = \mathbb{E}_t[\Phi_{t+1} u_{c,t+1}]$$

Debt as Random Walk

- Consider the FOC w.r.t b_{t+1}

$$\Phi_t \mathbb{E}_t[u_{c,t+1}] = \mathbb{E}_t[\Phi_{t+1} u_{c,t+1}]$$

$$\Rightarrow$$

$$\Phi_t = \mathbb{E}_t[\Phi_{t+1}] + \frac{COV_t[\Phi_{t+1}, u_{c,t+1}]}{\mathbb{E}_t[u_{c,t+1}]}$$

- Φ_t follows a risk-adjusted Martingale process and optimal allocation and policies are history dependent
- Under natural debt limit and quasi linear preferences $u(c,l) = u + v(l)$, Φ_t converges a.s. to first best (Doob Martingale Convergence Theorem)