Optimal Sustainable Intergenerational Insurance*

Francesco Lancia Alessia Russo Tim Worrall

March 2020 [Preliminary & Incomplete†]

Abstract

We examine intergenerational insurance in a stochastic overlapping generations endowment economy. Intergenerational insurance is sustainable when risk-sharing transfers are voluntary. Insurance transfers are chosen by a planner maximizing the discounted utility of all generations whilst respecting the limited enforcement of transfers. Optimal sustainable intergenerational insurance is history-dependent with occasional resetting. The risk from a temporary shock is spread into the future generating intergenerational consumption persistence. There is strong convergence to a non-degenerate invariant distribution. The theory can explain why a decrease of risk sharing over time and an impoverishment of younger generations compared to older ones is a sustainable and even optimal outcome.

Keywords: Intergenerational transfers; risk sharing; stochastic overlapping generations; limited commitment.

JEL CODES: D64; E21; H55.

*Lancia: Department of Economics, University of Salerno, Centre for Economic Policy Research and Center for Studies in Economics and Finance, Via Giovanni Paolo II, 132, 84084 - Fisciano (SA), Italy (e-mail: flancia@unisa.it). Russo: Department of Economics, BI Norwegian Business School and Centre for Economic Policy Research, Nydalsveien 37, N-0484 Oslo, Norway (e-mail: alessia.russo@bi.no). Worrall: School of Economics, University of Edinburgh, 30-31 Buccleuch Place, Edinburgh, EH8 9JT, UK (e-mail: tim.worrall@ed.ac.uk). We thank Spiros Bougheas, Francesco Caselli, Gabrielle Demange, Martin Gonzalez-Eiras, Paul Klein, Dirk Krueger, Sarolta Laczó, Espen Moen, Iacopo Morchio, Nicola Pavoni, José Víctor Ríos Rull, Kjetil Storesletten, and Aleh Tsyvinski for helpful comments. The paper has benefited from presentations at several conferences and workshops including at the NBER Summer Institute on Macro Public Finance, the SED Meeting in Edinburgh, the SAET Meeting in Faro, the CSEF-IGIER Symposium on Economics and Institutions at Anacapri, the CEPR European Summer Symposium in International Macroeconomics at Tarragona, the EEA-ESEM meetings in Manchester and the Vienna Macroeconomics Workshop. The third author acknowledges the support of ESRC grant ES/L009633/1.

†Please do not distribute or cite this version without permission.

1 Introduction

In many advanced economies, the living standard of the older generation has improved relative to the younger generation in recent decades.[1] The Great Recession and its aftermath has probably exacerbated this trend. If it persists, then it is likely that the younger generation will be less willing to provide the social insurance that the older generation expected. The prospect of such a potential intergenerational conflict is attracting attention of policy-makers to consider how risks should be allocated across different generations. How much should each cohort contribute to and receive from collective arrangements? How much risk should each generation bear?

In this paper, we address this question by requiring that any intergenerational insurance satisfies a participation constraint for the each generation. In particular, any transfers made by the young generation must be voluntary. This captures the idea that the young generation could renege on obligations to older generations if they are asked to contribute more than they expect to receive themselves when old.

To that aim, we consider a simple stochastic overlapping generations endowment economy with generational risk and limited enforcement of transfers. There is a representative agent in each generation and a single, non-storable consumption good. Agents live for two periods: young and old. The endowment of the young and old is stochastic, and the aggregate endowment might vary too, but the shock to endowments is identically and independently distributed over time. The are only two frictions in this economy. First, as is well-known (see, e.g., Peter Diamond, 1977; Robert C. Merton, 1983), in a stochastic overlapping generations model even if the economy is dynamically efficient, risk will, in general, not be allocated efficiently because there is no market in which the young can share risk with previous generations. Second, transfers between generations cannot be enforced. This limited enforcement may restrict the amount of risk that can be shared.

We suppose that the transfers between generations are decided by a benevolent social planner that maximizes the ex ante expected discounted utility of all generations. The planner respects the participation constraint of each generation. Since the old would not voluntarily make any transfer, insurance involves a transfer from the young to the old along with a promise to the young that they will receive a transfer when old depending on

[1]For example, the average disposable income of the U.S. young generation has fallen from just below the national average to just above 90% of the national average over the forty-year period 1974-2014. The older generation have fared much better over the same period. The average disposal income of the older generation has risen from approximately 70% of the national average to become roughly equal to the national average. Moreover, the older generation overtook the younger generation for the first time at about the time of the financial crisis. Data are from the Luxembourg Income Study (LIS) Database.

the endowment shocks. A plan of transfers thus, has the characteristics of a contingent pension plan. To be voluntary, a transfer from the young must at least match the lifetime utility from autarky given the transfers that generation is promised when old. That is, we assume that the penalty for reneging on a transfer is return to autarky. We call a plan of transfers sustainable if the participation constraints are satisfied for every generation.

It is a standard result that non-trivial sustainable transfers exist if endowments are such that the young wish to transfer consumption to old age at a zero net interest rate (Proposition 1). We are interested in cases where non-trivial sustainable transfers exist but where the first-best transfers cannot be sustained. We show that in this case, although the environment is stationary, the efficient sustainable intergenerational insurance is history dependent. To see this suppose that the ideal transfers would violate the participation constraint of a young generation in some state. To ensure the current transfer is voluntary, either the current transfer is reduced or the transfers promised to them when they are old are increased. Both changes are costly. Reducing the current transfer, reduces the current risk that is shared. Increasing the promised utility to the current young means tightening the participation constraints of the next generation. Since both changes are costly, there is an optimal trade-off between reducing the current transfer and increasing the future promised utility. Consequently, risk is partially spread across generations.

To see how the risk is spread consider some state of endowments where the future promise is increased. If the same state is repeated, then the next generation young are called upon to make a larger transfer and this in turn will required a higher future promise to them as well. The efficient solution is history dependent because the transfer made by the young depends not only on the endowment state but on the previous promise. However, as the promised utility to the next generation is increased, there will be other endowment states where a high transfer is undesirable and it is better to reduce the current transfer and the future promised utility. That is, the promised utility will increase or decrease depending on the history of states. We show (Lemma 2) that, when the promised utility is low enough, there is always some state where the participation constraint of the young is not binding. In that case, the future promise is *reset* to the value that maximizes the payoff of the planner, a value that is independent of state or history.

Even though the efficient sustainable intergenerational insurance is history dependent, we use the resetting property to show that there is strong convergence to a unique invariant distribution of promised utilities (Proposition 4). The invariant distribution of promised utility is non-degenerate but the distribution of consumption across and between generations will evolve dynamically over time. This is in stark contrast to the situation if either transfers on the young can be enforced or there is no risk. In the former case, the promised utility is constant over time, except possibly at the initial date (Proposition 2). In the latter case, promised utility is constant in the long run, though there can be a finite initial phase where the promised utility is declining (Proposition 3). Thus, both risk and limited enforcement are required to have dynamics of consumption in the long run.

To quantify how risk is spread across and between generations and in the long run we consider the state prices implied by the solution for consumption. In particular, we use the entropy measure used by David Backus, Mikhail Chernov and Stanley Zin (2014) and the metric used by Ian Martin and Stephen Ross (2019). Compared to other risk measures, the entropy measure has the advantage to quantify the risk aversion reflected in bond prices in a way that captures the dynamics of consumption. This is an important feature in our context since in the constrained efficient allocation the risk from a temporary shock in any period is spread into the future generating a persistent time variation in the expected consumption growth.

We also provide a detailed analysis of the case where there are just two endowment shocks. In a special case, we fully characterize the invariant distribution of promised utility and provide a closed form solution for the risk measure (Proposition 6). The consumption path follows a simple rule with resetting to the value of promise that maximizes the payoff of the planner any time that an unfavorable state for the young occurs. In this case, the invariant distribution of promised utility is a countable geometric distribution, consumption of the young is positively autocorrelated, and the short-run risk is smaller that long-run risk faced by generations.

The model itself is entirely stationary and is the simplest possible to address intergenerational insurance with limited enforcement. We show however that the analysis of the optimal sustainable insurance can be easily extended in a number of relevant directions without substantially modifying any of the results. In particular, we incorporate into the model preference heterogeneity between young and old in order to emphasize the role of intergenerational conflicts, as well as warm-glow preferences in order to shed the light on the role of altruism; exogenous income growth and a storage technology in order to show that under some conditions the stationary setting we are considering is without loss of generality.

The paper contributes to three main areas of the literature. First, it contributes to the literature on risk sharing and limited commitment. That literature has been concerned with the case of infinitely-lived agents. It has focused on two polar cases: either with two infinitely-lived agents (see. e.g., Jonathan Thomas and Tim Worrall, 1988; Narayana Kocherlakota, 1996) or with a continuum of infinitely-lived agents (see, e.g., Jonathan Thomas and Tim Worrall, 2007; Dirk Krueger and Fabrizio Perri, 2011; Tobias Broer, 2013). Our overlapping generations model has a continuum of agents but only two agents are alive at any time and fills an important gap in the existing literature. We discuss the relationship of our results to the previous literature in Section 9.

Secondly, it contributes to the large literature on risk sharing in overlapping generation models. Many papers consider public policies or other non-market mechanisms that improve risk sharing through social security (see, e.g., Walter Enders and Harvey Lapan, 1982; Robert J. Shiller, 1999; Antonio Rangel and Richard Zeckhauser, 2000). However, this literature almost exclusively focuses on mandatory transfers and restricts attention to stationary transfers in contrast to the current paper that has voluntary and history-dependent transfers. Our result is foreshadowed by Roger Gordon and Hal Varian (1988) who established in a mean-variance setting that any time-consistent optimal intergenerational risk-sharing agreement cannot be stationary. The paper is also related to Laurence Ball and Gregory Mankiw (2007) who consider a complete markets equilibrium in which all generations can trade. Unlike our paper, they consider a case where either risk occurs only in one period or where risk is always small.

A number of papers that examine stochastic overlapping generations models aim to provide a simple necessary and sufficient criterion for Pareto-optimality. S. Rao Aiyagari and Dan Peled (1991) provide a simple necessary and sufficient condition for ex interim optimality in an endowment economy with a finite state space in terms of a dominant root condition. This approach has been extended by a number of authors (see, e.g., Rodolfo Manuelli, 1990; Subir Chattopadhyay and Piero Gottardi, 1999; Gabrielle Demange and Guy Laroque, 1999; Gaetano Bloise and Filippo L. Calciano, 2008). Pamela Labadie (2004) shows how to interpret this characterization in terms of ex ante Pareto-optimality. We will show that an appropriately modified condition also applies in our environment with limited commitment and we discuss this further in Section 9.

Thirdly, our approach is related to the literature on the political economy of social security. Typically, this literature (see, e.g., Thomas Cooley and Jorge Soares, 1999; Costas Azariadis and Vincenzo Galasso, 2002; Antonio Rangel, 2003; Gabrielle Demange, 2009) analyzes settings in which intergenerational transfers are decided by majority rule. In this case, social security is supported because the young, who are in the majority vote for social security in the expectation that the next generation will do the same. The resulting equilibria are not necessarily Pareto optimal. In contrast, our approach characterizes constrained efficient intergenerational transfers in which each generation unanimously agrees to participate whilst respecting promises previously made.

The paper is organized as follows. Section 2 sets out the model. Section 3 considers two benchmark models: one with full enforcement and one with no risk. Section 4 provides the main results of the paper and Section 5 shows that there is convergence to an invariant distribution in the long run. Section 6 considers how risk is allocated in the invariant distribution. Section 7 studies the case with two endowment shocks. Since the main model of the paper is deliberately simple Section 8 considers its robustness to a number of possible extensions. Section 9 discusses aspects of the model and its relationship to some of the relevant literature. Section 10 concludes. Proofs are found in the Appendix. Additional proofs and further details are contained in the Supplementary Appendix.

2 The Model

Time is discrete and indexed by $t = 0, 1, 2, \ldots, \infty$. The model consists of a pure exchange economy with an overlapping generations demographic structure. At each time t a new generation is born and lives for two periods. Each generation is composed of a single agent.[2] The agent is young in the first period of life and old in the second period. The economy starts at date $t = 0$ with an initial old and initial young generation. Since time is infinite, the initial old is the only generation that lives for just one period.

At each time t, agents receive an endowment of a perishable consumption good that depends on the state of the world $s_t \in \mathcal{S} := \{1, 2, \ldots, S\}$ with $S \geq 2$. Denote the history of states up to and including date t by $s^t := (s_0, s_1, ..., s_t) \in \mathcal{S}^t$ and the probability of reaching history s^t by $\pi(s^t)$ where $\pi(s^{t-1})\pi(s_t \mid s^{t-1}) = \pi(s^{t-1}, s_t)$. We assume that states are identically and independently distributed (hereafter, i.i.d.). Hence, $\pi(s^t) = \pi(s_0) \cdot \ldots \cdot \pi(s_t)$ where $\pi(s_t)$ is the probability of state s_t and $\pi(s_{t+1} \mid s^t) = \pi(s_{t+1})$. Given s_t, endowments of the young and old agents are $e^y(s_t)$ and $e^o(s_t)$ and the aggregate endowment is $e(s_t) := e^y(s_t) + e^o(s_t)$. Endowments are finite and strictly positive. All information about endowments is public: there is complete information. There is no technology for saving or investment. Hence, the aggregate endowment must be consumed within the

[2] The assumption that there is a representative agent in each generation is to focus on intergenerational risk sharing. By doing so, however, we ignore questions about inequality within generations and its evolution over time.

period. Let $c^y(s^t)$ and $c^o(s^t)$ be the per-period individual consumption of the young and old generation. Since the endowment will not be wasted we have $c^y(s^t) + c^o(s^t) = e(s_t)$. Note that endowments depend only on the current state whereas consumption can, in principle, depend on the history of states. In autarky, agents consume just their own endowments, that is, $c^y(s^{t-1}, s_t) = e^y(s_t)$ and $c^o(s^{t-1}, s_t) = e^o(s_t)$ for all t and (s^{t-1}, s_t).

Each generation is born after that period's uncertainty is resolved and when current endowments are known. After its birth, therefore, a generation faces uncertainty only in old age. This means that there cannot exist any insurance market to insure the young against endowment risk. The lifetime endowment utility of an agent born in state s_t is:

$$\hat{v}(s_t) := u(e^y(s_t)) + \beta \sum_{s_{t+1}} \pi(s_{t+1}) u(e^o(s_{t+1}))$$

where $\beta \in (0, 1]$ is a common generational discount factor and $u(\cdot)$ is the utility function common to both young and old.

Assumption 1 *The utility function $u\colon \mathbb{R}_+ \to \mathbb{R} \cup \{-\infty\}$ is strictly increasing, strictly concave, thrice continuously differentiable and satisfies the Inada condition. In addition*

$$u(0) < \min_r \left\{ u(e^y(r)) - \beta \sum_s \pi(s) \left(u(e(s)) - u(e^o(s)) \right) \right\}.$$

The latter part of this assumption is sufficient to guarantee that the intergenerational insurance we consider has non-negative consumption for each agent. Obviously, this condition is satisfied when $\lim_{c \to 0} u(c) = -\infty$, as is the case for the logarithmic utility function or CRRA utility functions with a coefficient of risk aversion greater than one. Since endowments are positive and finite, $\hat{v}(s_t)$ is bounded.

Let $\lambda(s_t)$ denote the ratio of marginal utilities of the young and old at the autarky:

$$\lambda(s_t) := \frac{u_c(e^y(s_t))}{u_c(e^o(s_t))}.$$

If $\lambda(s_t) > 1$, then it follows, from the assumption that the utility of consumption is independent of age, that $e^y(s_t) < e^o(s_t)$ and the old are wealthier in that state. States are ordered such that $\lambda(S) \geq \lambda(S-1) \geq \cdots \geq \lambda(1)$. Thus, a higher state indicates that the old are relatively wealthier than the young than in a lower state.[3] Since $\lambda(s_t)$ varies

[3] Where two states have the same value of $\lambda(\cdot)$, we will use the convention that the states are ordered by the aggregate endowment, that is, higher states associated with higher aggregate endowment. A special case is where states can be ordered so that the endowment of the old is increasing in the state

across states, it is desirable to share risk across generations. In the absence of a storage technology and because the young are born after the state is resolved, the only possibility for intergenerational insurance is via transfers between young and old generations.[4] We assume, however, that all transfers must be voluntary or self-enforcing. That is, agents will make a transfer only if making a transfer is in their interest. We assume that any generation that does not make a transfer when called upon to do so, is punished by receiving no transfer when they reach old age. Therefore, any intergenerational insurance must provide all generations with at least the same lifetime utility they derive from the autarkic consumption, for any possible history of shocks.[5]

Benevolent Social Planner Consider the problem of a benevolent social planner who chooses intergenerational insurance. An intergenerational insurance rule is a function $\tau(s^t)$ that specifies the transfer between the young and the old for any possible history s^t. Since the aggregate endowment will be consumed, we can write:

$$\tau(s^t) = e^y(s_t) - c^y(s^t) \quad \text{and} \quad c^o(s^t) = e(s_t) - c^y(s^t). \tag{1}$$

The planner must respect the constraints that neither the old nor the young generation would be better off at autarky than in an allocation with intergenerational transfers. For the old generation this implies that the transfer must always be non-negative because if the old were ever called upon to make a transfer, they would default. Expressed in terms of the consumption of the young this means:

$$c^y(s^t) \leq e^y(s_t) \qquad \forall\, s^t. \tag{2}$$

Since consumption is required to be non-negative, constraint (2) can be expressed by requiring $c^y(s^t) \in \mathcal{Y}(s_t) := [0, e^y(s_t)]$ for each history s^t. For the young generation, the analogous participation constraint requires that any transfer made when young is compensated enough by transfers received when old to be no worse off than reneging on the transfer today and receiving the corresponding autarky utility:

$$u(c^y(s^t)) + \beta \sum_{s_{t+1}} \pi(s_{t+1}) u(e(s_{t+1}) - c^y(s^t, s_{t+1})) \geq \hat{v}(s_t) \qquad \forall\, s^t. \tag{3}$$

Let $\Lambda := \{\{c^y(s^t) \in \mathcal{Y}(s_t)\}_{t=0}^{\infty} \mid (3)\}$ denote the constraint set for the planner.

Definition 1 (Sustainability) *An Intergenerational Insurance rule is sustainable if the*

and the endowment of the young is decreasing in the state.

[4] We ignore any altruistic or bequest motives for making transfers.

[5] The assumption that the transfer is voluntary can be interpreted as a plan that requires support of both generations if put to a vote.

history-dependent sequence $\{c^y(s^t)\}_{t=0}^{\infty} \in \Lambda$.

Since utility is strictly concave and constraints (3) are linear in utility, the constraint set Λ is convex and compact.

The social planner seeks to address the conflict between generations by choosing a sustainable intergenerational insurance rule to maximize the discounted sum of expected utilities of all generations. We suppose that the planner discounts the expected utility of future generations with a social discount factor $\delta \in (0,1)$ and weighs the utility of the initial old by β/δ.[6]

Definition 2 (Optimality) *A Sustainable Intergenerational Insurance rule* $\{c^y(s^t)\}_{t=0}^{\infty} \in \Lambda$ *is optimal if it maximizes a weighted sum of expected utilities of all the generations, that is, maximizes*

$$\frac{\beta}{\delta}\sum_{s_0}\pi(s_0)u(e(s_0)-c^y(s_0))+\mathbb{E}_0\left[\sum_t \delta^t\left(u(c^y(s^t))+\beta\sum_{s_{t+1}}\pi(s_{t+1})u(e(s_{t+1})-c^y(s^t,s_{t+1}))\right)\right], \tag{4}$$

where $\mathbb{E}_0$ *is the expectation over all future histories, subject to the constraint*

$$\sum_{s_0}\pi(s_0)u(e(s_0)-c^y(s_0)) \geq \omega, \tag{5}$$

for some given ω.

Let $V(\omega)$ denote the value function corresponding to a solution of the optimization problem in Definition 2. The function $V(\omega)$ traces out the Pareto-frontier between the expected utility of the current old and the expected discounted utility of future generations.[7] The function $V(\omega)$ is defined on a set $\Omega = [\omega_{\min}, \omega_{\max}]$, where $\omega_{\min} := \sum_{s_t}\pi(s_t)u(e^o(s_t))$ is the expected autarky utility of the old and $\omega_{\max}$ is the highest feasible value of expected utility of the old consistent with the participation constraints (it will be defined more explicitly below). The constraint (5) requires that the planner, in making the choice at date $t=0$ before the state s_0 is known, chooses a level of $c^y(s_0)$ such that expected utility

[6] A planner's geometric discounting is common in the literature (see, e.g., Emmanuel Farhi and Iván Werning, 2007). Giving a weight of β/δ to the initial old means that the relative social weights on the old and young are the same at every time period.

[7] More precisely, the function $\tilde{V}(\omega) := V(\omega) - \omega$ can be considered as a Pareto frontier that trades off the expected utility of the current old against the planner's valuation of the expected discounted utility of all other generations.

offered to the initial old generation is of at least ω. Let

$$\omega_0 := \sum_{s_0} \pi(s_0) u(e(s_0) - \tilde{c}^y(s_0)), \tag{6}$$

where the notation $\tilde{c}^y(s_0)$ is adopted here to emphasize that it is part of the optimal solution to the problem in Definition 2. Clearly, for $\omega \leq \omega_0$, constraint (5) does not bind and $V(\omega) = V(\omega_0)$; whereas for $\omega > \omega_0$, constraint (5) binds and $V(\omega) < V(\omega_0)$. Attention can be restricted to $\omega \geq \omega_0$, because if $\omega < \omega_0$, then the consumption choice for the initial old is the same with or without constraint (5). We will also show in more details below that there may be some high values of ω that are transitory. In this case, we denote the upper bound of the long run values by $\bar{\omega}$ and in the long run attention can be restricted to the set $[\omega_0, \bar{\omega}] \subseteq \Omega$.

Preliminaries The existence of a sustainable non-autarkic allocation can be addressed by considering small stationary (depending only on the current state) transfers. Denote the intertemporal marginal rate of substitution of consumption when young in state s_t for consumption when old in state s_{t+1}, evaluated at the autarky, by $\hat{m}(s_t, s_{t+1}) := \beta u_c(e^o(s_{t+1}))/u_c(e^y(s_t))$ and let $\hat{q}(s_t, s_{t+1}) := \pi(s_{t+1})\hat{m}(s_t, s_{t+1})$. The terms $\hat{m}(s_t, s_{t+1})$ and $\hat{q}(s_t, s_{t+1})$ correspond to the pricing kernel and state prices in an equilibrium model. Denote the matrix of terms $\hat{q}(s_t, s_{t+1})$ by $\hat{Q}$. It is well known that autarky is not Pareto optimal whenever the Perron root, that is, the largest eigenvalue of the matrix $\hat{Q}$, is greater than one (see, e.g., Aiyagari and Peled, 1991; Chattopadhyay and Gottardi, 1999). Hence, there exist non-negative stationary transfers that improve lifetime utility of the young in each state. A simple sufficient condition for this to be true is that the Frobenius lower bound, given by the minimum row sum of $\hat{Q}$, is greater than one. We shall assume this from now on.[8]

Assumption 2 *For each state $s_t \in \mathcal{S}$,*

$$\sum_{s_{t+1}} \hat{q}(s_t, s_{t+1}) = \beta \sum_{s_{t+1}} \pi(s_{t+1}) \frac{u_c(e^o(s_{t+1}))}{u_c(e^y(s_t))} > 1.$$

This assumption can be interpreted as saying that at the autarky allocation and in every state the young would, if they could, prefer to save for their old age, even at a zero rate of interest. If there were just one state, Assumption 2 reduces to the standard Samuelsonian condition, $\beta u_c(e^o) > u_c(e^y)$. In this case, where the young are relatively wealthy, it is well

[8]Since Assumption 2 is a sufficient condition for the Perron root to be greater than one, it is stronger than needed to ensure the existence of a sustainable allocation. For example, in the two state case we consider in Section 7, the required existence condition is $\hat{q}(1,1) + \hat{q}(2,2) > 1$.

known (Paul Samuelson, 1958) that there are Pareto-improving transfers from young to old. In the stochastic case, Assumption 2 implies that a sufficient condition for Pareto-improving transfers is that young should be on average wealthier than old. This is a natural assumption given that our focus is on transfers to the old. Following Aiyagari and Peled (1991); Chattopadhyay and Gottardi (1999), we have that the constraint set Λ is non-empty (a simple proof of which is given in the Supplementary Appendix).

Proposition 1 *Under Assumption 2, there is a non-trivial and stationary sustainable Intergenerational Insurance rule.*

It can be shown that Assumption 2 implies $\beta u_c(e^o(1)) > u_c(e^y(1))$, or equivalently $\lambda(1) < \beta$. That is, there is at least one state in which the young are relatively wealthier than the old. We shall also make the converse assumption that there is at least one state in which the old are relatively wealthier than the young. In particular,

Assumption 3

$$\lambda(S) \geq \frac{\beta}{\delta}.$$

Since $\delta < 1$, this inequality implies $\lambda(S) > \beta$, or equivalently $\beta u_c(e^o(S)) < u_c(e^y(S))$. We make Assumption 3 for two reasons. First, because it shows that our results do not depend on the standard Samuelsonian condition applying in every state. Second, because it provides a simple sufficient condition for the strong convergence result we present in Section 5.

We will shortly turn to the characterization of optimal intergenerational insurance. Before doing so, it is useful to consider two benchmarks, which will serve to illustrate the inefficiencies generated by the presence of limited enforcement and uncertainty.

3 Two benchmark cases

In this section, we characterize the intergenerational insurance rule chosen by a benevolent social planner in two benchmark cases. In the first, the planner can enforce transfers from the young but not from the old. That is, the planner respects constraint (2) of the old but can ignore constraint (3) of the young. We will refer to this as the first-best outcome. The second benchmark has no uncertainty but requires that the planner respects both the constraints of the old and the young.

First Best

In the first benchmark, we assume that there is uncertainty, $S \geq 2$, but ignore the participation constraint of the young. That is, we assume the planner can fully enforce transfers from the young but cannot enforce transfers from the old.[9] Let $\Lambda^* := \{c^y(s^t) \in \mathcal{Y}(s_t)\}_{t=0}^{\infty}$ denote the set of transfers ignoring the constraints of the young. Hereafter, the asterisk designates the first-best outcome.

Definition 3 (First Best) *An Intergenerational Insurance* $\{c^y(s^t)\}_{t=0}^{\infty} \in \Lambda^*$ *is first best if it maximizes the objective function* (4) *subject to constraint* (5).

It is easy to check that the first-best $c^{y*}(s^t)$ satisfy the following first-order condition:

$$\frac{u_c(c^{y*}(s^t))}{u_c(e(s_t) - c^{y*}(s^t))} = \max\left\{\frac{\beta}{\delta}, \lambda(s_t)\right\} \qquad \forall\, s^t. \tag{7}$$

Condition (7) requires that in the absence of the young participation constraint, $c^{y*}(s^t)$ is independent of the history s^{t-1} and depends only on the current state s_t. Then, transfers are stationary. Let $\tau^*(s) = e^y(s) - c^{y*}(s)$ denote the first-best transfer in state s. For states in which the old participation constraint is tight, namely when $\beta/\delta \leq \lambda(s)$, $\tau^*(s) = 0$. Under Assumption 3, there always exist such states and hence, the first-best transfer is not positive in every state.[10] Transfers to the old are made only in states where $\beta/\delta > \lambda(s)$. For this set of states, condition (7) means that risk is shared efficiently across the generations. It is the familiar optimal risk sharing condition from Karl Borch (1962), modified to account for the constraint that transfers are only from young to old and is designated by Labadie (2004) as an *equal-treatment Pareto optimum* where all generations are weighted equally by the planner. It can be seen from condition (7) that for states where transfers are positive $\tau^*(s)$ is increasing in β, because a higher β means more weight is placed on the utility of the old to whom the transfer is made, whereas the transfer is decreasing in δ, because a higher δ means the planner puts more weight on the utility of the young who make the transfer.[11]

Let $\omega^* := \sum_s \pi(s) u(e(s) - c^{y*}(s))$ denote the expected utility of the old at the first-best solution. From the definition in (6) it follows that $\omega_0 = \omega^*$. Now consider constraint (5).

[9] We could define the first-best to allow the planner to enforce transfers from the old. The reason for presenting the first best as we do is to show that the optimal allocation is stationary and therefore any history-dependence in transfers in the optimal sustainable intergenerational insurance (Definition 2) derives from the imposition of the participation constraints of the young.

[10] If $\lambda(s) = \beta/\delta$, then $\tau^*(s) = 0$: the old participation constraint holds with equality but is not binding.

[11] The set of states with not binding participation constraints of the old is weakly increasing in β and weakly decreasing in δ.

For $\omega \leq \omega^*$, constraint (5) does not bind and the optimum consumption $c^{y*}(s_0)$ is determined by condition (7), as for every other period $t > 0$. If $\omega > \omega^*$, then constraint (5) binds and the initial transfers to the old are correspondingly higher whilst maintaining a constant ratio of marginal utilities across states. In particular, if $\omega > \omega^*$ then there is a $\nu_0 > 0$ such that the consumption at date $t = 0$ satisfies:

$$\frac{u_c(c^{y*}(s_0))}{u_c(e(s_0) - c^{y*}(s_0))} = \max\left\{\frac{\beta}{\delta}(1 + \nu_0), \lambda(s_0)\right\} \qquad \forall\, s_0, \tag{8}$$

and the constraint (5) holds with equality.[12]

Let denote the per-period payoff to the planner with the first-best transfers by $v^* := \sum_s \pi(s)u(c^{y*}(s)) + (\beta/\delta)\omega^*$ and the expected discounted payoff of the planner for $\omega \in \Omega$ by $V^*(\omega)$. The first-best outcome that respects the participation constraint of the old is summarized in the following proposition.[13]

Proposition 2 *(i) The consumption of the young $c^{y*}(s^t)$ is stationary and satisfies condition (7) for all $t > 0$ and condition (8) for $t = 0$; (ii) The function $V^*\colon \Omega \to \mathbb{R}$ is equal to $V^*(\omega) = v^*/(1-\delta)$ for $\omega \in [\omega_{\min}, \omega^*]$ and is strictly decreasing and strictly concave for $\omega \in (\omega^*, \omega_{\max}]$ with $\lim_{\omega\to\omega_{\max}} V^*_\omega(\omega) = -\infty$.*

Note that when constraint (5) is imposed and $\omega > \omega^*$, the consumption of the old generation is higher than the first best but only for the initial period. There is immediate convergence to the stationary distribution with a single mass point at $\{\omega^*\}$ after the first period.

Deterministic Case

We now consider the case in which there is only one state but in which any intergenerational transfer respects the participation constraint of the young as well as the old. In the case of one state, Assumption 2 reduces to the standard Samuelsonian condition: $\beta > \lambda$ with $\lambda := u_c(e^y)/u_c(e^o)$. This assumption together with the strict concavity of the utility function means that there is a unique consumption level $c^y_{\min} < e^y$ equal to the smallest *stationary* consumption of the young that satisfies the the participation constraint

$$u(c^y_{\min}) + \beta u(e - c^y_{\min}) = \hat{v} := u(e^y) + \beta u(e^o). \tag{9}$$

[12]In equation (8), $(\beta/\delta)\nu_0$ is the multiplier associated with constraint (5).
[13]The proof and properties of the value function are standard and easily checked.

The corresponding old utility is $\omega_{\max} = u(e - c^y_{\min})$. From the previous benchmark, the first-best c^{y*} satisfies $u_c(c^{y*})/u_c(e - c^{y*}) = \max\{\beta/\delta, \lambda\}$ and the corresponding utility of the old is $\omega^* = u(e - c^{y*})$. Since $\beta/\delta > \beta > \lambda$, the participation constraint of the old is satisfied at c^{y*}. Whether the participation constraint of the young is satisfied at c^{y*} depends on the value of δ. For δ above some critical value, $c^{y*} > c^y_{\min}$ and the first-best consumption is sustainable, that is, it satisfies the participation constraint of the young. For δ below this critical value, $c^{y*} < c^y_{\min}$ and the first-best transfer is not sustainable.

Let c^y_t denote the consumption of the young at date t and let ω_t denote the corresponding utility of the old. Consider the solution to the maximization problem in (4) with the young participation constraints $u(c^y_t) + \beta u(e - c^y_{t+1}) \geq \hat{v}$ for $t \geq 0$. The solution to this problem is $c^y_t = \max\{c^{y*}, c^y_{\min}\}$ for all $t \geq 0$. Now consider constraint (5). If $\omega \leq \omega^*$, then it will be optimal to set consumption $c^y_t = \max\{c^{y*}, c^y_{\min}\}$ for all $t \geq 0$. On the other hand, consider a case where δ is large enough such that $c^{y*} > c^y_{\min}$ and suppose $\omega \in (\omega^*, \omega_{\max})$. At date $t = 0$, c^y_0 must satisfy $u(e - c^y_0) \geq \omega$. This requires $c^y_0 < c^{y*}$. It is clear that is desirable to set c^y_0 such that $u(e - c^y_0) = \omega$ and $c^y_1 = c^{y*}$. However, setting $c^y_1 = c^{y*}$ can violate the participation constraint of the young. In such a case, c^y_1 has to be chosen from $u(c^y_0) + \beta u(e - c^y_1) = \hat{v}$, which implies that $c^y_1 < c^{y*}$. Repeating this argument for $t > 1$ shows that given c^y_t, c^y_{t+1} either satisfies $u(c^y_t) + \beta u(e - c^y_{t+1}) = \hat{v}$ or $c^y_{t+1} = c^{y*}$ if $u(c^y_t) + \beta u(e - c^{y*}) \geq \hat{v}$. It is useful to express this rule in terms of a policy function that gives the next period value of the utility of the old, ω_{t+1}, as a function of the current value ω_t, namely,

$$\omega_{t+1} = \begin{cases} \omega^* & \text{for } \omega_t \in [\omega_{\min}, \omega^c], \\ \frac{1}{\beta}\left(\hat{v} - u\left(e - u^{-1}(\omega_t)\right)\right) & \text{for } \omega_t \in (\omega^c, \omega_{\max}], \end{cases} \tag{10}$$

where $\omega_{\min} = u(e^o)$ and $\omega^c := u(e - u^{-1}(\hat{v} - \beta\omega^*))$. It follows from the strict concavity of the utility function that $\omega^c > \omega^*$. The policy function is illustrated in Figure 1. It is increasing and convex in ω_t. The dynamic evolution of ω_t is straightforward. For $\omega_t \in [\omega_{\min}, \omega^c]$, $\omega_{t+1} = \omega^*$ for each $t \geq 0$. For $\omega_t \in (\omega^*, \omega_{\max}]$, ω_{t+1} declines monotonically. Since $\omega^c > \omega^*$, the process for ω_t converges to the long-run level in finite time.

Let denote the per-period payoff to the planner with the first-best allocations without uncertainty by $v^* := u(c^{y*}) + (\beta/\delta)\omega^*$ and the expected discounted payoff of the planner for $\omega_t \in \Omega$ by $V(\omega_t)$. The solution for the deterministic case with sustainable ω^* is summarized in the following proposition.[14]

Proposition 3 *(i) If $\omega \in [\omega_{\min}, \omega^*]$, the consumption of the young is $c^y_t = c^{y*}$ for all $t \geq 0$*

[14]A proof of Proposition 3 is provided in the Supplementary Appendix.

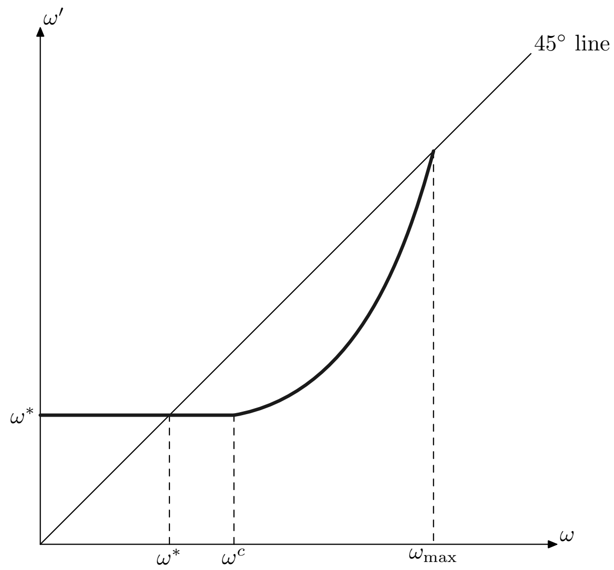

Figure 1: Policy Function in the Deterministic Case.

where c^{y*} satisfies $u_c(c^{y*})/u_c(e-c^{y*}) = \beta/\delta$. (ii) If $\omega \in (\omega^*, \omega_{\max}]$, the consumption of the young c_t^y and corresponding utility of the old ω_{t+1} satisfy the policy function (10)*. There exists a finite time $\hat{t}$ such that c_t^y is monotonically increasing for $t < \hat{t}$ and $c_t^y = c^{y*}$ for all $t \geq \hat{t}$. (iii) The function $V\colon \Omega \to \mathbb{R}$ is equal to $V(\omega) = \upsilon^*/(1-\delta)$ for $\omega \in [\omega_{\min}, \omega^*]$ and is strictly decreasing and strictly concave for $\omega \in (\omega^*, \omega_{\max}]$ with $\lim_{\omega \to \omega_{\max}} V_\omega(\omega) = -\infty$.*

The solution is either stationary or converges monotonically to a stationary point where $c_t^y = c^{y*}$ in a finite time. Hence, the long-run distribution of ω is degenerate. In particular, for the case where $c^{y*} > c_{\min}^y$, it has a single mass point at $\{\omega^*\}$.

In the next sections, we shall show that with both uncertainty and limited enforcement, the long run distribution for ω is non-degenerate. The benchmarks show that it is the combination of limited enforcement and uncertainty that is necessary for this result.

4 Optimal Policy Functions

In this section, we characterize the optimal intergenerational insurance under limited commitment when the planner respects the participation constraints of both the young and the old. To proceed we rule out the case where the first-best outcome can be sustained in all states. In particular, we consider the case where the first-best transfer will violate the participation constraint of the young in at least one state.[15]

Assumption 4 *There is at least one state $s \in \mathcal{S}$ such that*

$$u(c^{y*}(s)) + \beta \sum_s \pi(s) u(e(s) - c^{y*}(s)) < \hat{v}(s).$$

We reformulate the optimization problem described in Definition 2 recursively. This can be done because states are i.i.d. and all constraints are forward looking. Our characterization is similar to the promised-utility approach developed by Thomas and Worrall (1988), amongst others. For simplicity of notation we often omit t indexes and use primes to denote next-period variables. At each time t, the expected utility ω promised to the current old is the state variable, which encompasses information about the past history. It must belong to the interval Ω. The problem at each date t is to determine state contingent young consumption $c^y(s)$ and the state contingent expected old-age utility that is promised to current young, which we denote by $\omega'(s)$.

The participation constraint for the young, that is, constraint (3), can be rewritten as follows:

$$u(c^y(s)) + \beta \omega'(s) \geq \hat{v}(s) \qquad \forall\, s \in \mathcal{S}. \tag{11}$$

The participation constraint for the old is subsumed in the requirement that $c^y(s) \in \mathcal{Y}(s)$ for all s. In each period, the utility promises made to the current generations must be feasible, that is,

$$\omega'(s) \leq \omega_{\max} \qquad \forall\, s \in \mathcal{S}; \tag{12}$$

and the expected utility given to the current old generation must be at least that previously promised:

$$\sum_s \pi(s) u(e(s) - c^y(s)) \geq \omega. \tag{13}$$

Constraint (13) is a promise-keeping constraint and is analogous to constraint (5), but it is now required to hold in every period.[16] Define Γ as the set of state contingent young consumption $c^y(s)$ and feasible state contingent promises $\omega'(s)$ to the current young such that constraints (11), (12) and (13) are satisfied: $\Gamma := \{\{c^y(s) \in \mathcal{Y}(s), \omega'(s) \in \Omega\}_{s \in \mathcal{S}} \mid (11), (12) \text{ and } (13)\}$. It is easily checked that the set Γ is compact and convex because the utility function is strictly concave and Ω is an interval. The value function $V(\omega)$ satisfies

[15] We show by an example, given below, that there is a non-empty set of parameter values such that Assumptions 2, 3 and 4 are simultaneously satisfied.

[16] Note that $u^{-1}(\omega)$ is the promise offered to the old expressed in certainty equivalent consumption.

the following functional equation:

$$V(\omega) = \max_{\{c^y(s),\omega'(s)\}_s \in \Gamma} \left[\sum_s \pi(s) \left(\tfrac{\beta}{\delta} u(e(s) - c^y(s)) + u(c^y(s)) + \delta V(\omega'(s)) \right) \right]. \tag{14}$$

Let denote the state vector by $x := (\omega, s)$ and (with a slight abuse of notation) the solution of the maximization problem for c^y and ω' by the two stochastic policy functions $c^y(x)$ and $f(x)$. The function $c^y(x)$ is the optimum consumption of the current young when the current old have a promised expected utility of ω and the current state is s. Likewise, $f(x)$ is the expected utility promise given to the current young. The optimal allocation can be computed recursively by starting with some $\omega_0 \in \Omega$, and solving for the transfers in the initial period together with future promised utilities in each possible state.[17] The future promised utilities are then used to solve the dynamic program for the next period and so on. The existence of the optimal allocation is guaranteed from the compactness of Ω and Γ. Uniqueness is guaranteed by the strict concavity of $u(\cdot)$.

The function $V(\omega)$ cannot be found by a standard contraction mapping argument starting from an arbitrary value function because the value function associated with the autarky allocation also satisfies the functional equation (14). Nevertheless, a similar approach can be used to iterate on the value function starting from the first-best value function derived by ignoring the participation constraints of the young. Following the arguments of Jonathan, Thomas and Tim Worrall (1994), it can be shown that the limit of this iterative mapping is the optimal value function $V(\omega)$.[18] Proposition 2 established that the first-best value function is non-increasing, differentiable and concave and the limit value function inherits these properties. In particular, we have:

Lemma 1 *The function $V\colon \Omega \to \mathbb{R}$ is non-increasing, continuously differentiable and concave in ω, where $\Omega = [\omega_{\min}, \omega_{\max}]$ and $\omega_{\min} < \omega_{\max} < \sum_s \pi(s) u(e(s))$. There is an $\omega_0 \in (\omega_{\min}, \omega^*)$ such that $V(\omega)$ is constant for $\omega \leq \omega_0$, is strictly decreasing and strictly concave for $\omega > \omega_0$ with $V_\omega(\omega_0) = 0$ and $\lim_{\omega \to \omega_{\max}} V_\omega(\omega) = -\frac{\beta}{\delta}\bar{\nu}$, where $\bar{\nu} \in \mathbb{R}_+ \cup \{\infty\}$.*

Concavity of the value function follows from the concavity of the objective and convexity of the constraint set. The lower endpoint $\omega_{\min}$ of the domain of $V(\omega)$ is the autarky value because zero transfers are feasible. The upper endpoint $\omega_{\max}$ is determined by the choice of young consumption and promises that maximize the expected utility of the current old subject to the constraints (11) and (12). This is itself a strictly concave programming

[17] We discuss how ω_0 is determined after Lemma 1.

[18] Indeed, it is precisely this iteration starting from the first-best value function that is used to compute the optimal value function when we solve numerical examples in Section 7.

problem and has a unique solution.[19] The last part of Assumption 1 is sufficient to guarantee $\omega_{\max} < \sum_s \pi(s)u(e(s))$. Differentiability follows because the constraint set satisfies a linear independence constraint qualification when $\omega \in [\omega_{\min}, \omega_{\max})$. The left hand derivative of $V(\omega)$ evaluated at $\omega_{\max}$ may be finite or infinite. Whether the derivative is finite or infinite depends on whether, or not, $\omega_{\max}$ is part of the ergodic set, and is discussed in more detail below.[20]

Given the properties of the value function described in Lemma 1, the planner will choose an initial promise ω_0 (hence the notation), because ω_0 is the largest promised utility that maximizes the payoff of the planner. That is, for $\omega < \omega_0$, the promise to the initial old can be increased toward the first-best without reducing the payoff of the planner. Thus, we can restrict attention to promises $\omega \geq \omega_0$ in what follows. This initial promise will be determined as part of the solution and will depend, in general, on all parameter values. Assumption 4 however, means that the first-best violates one of the participation constraints in (11) and hence $\omega_0 < \omega^*$.

We now turn to the properties of the policy functions $c^y(x)$ and $f(x)$. Given the differentiability of the value function, the first-order conditions for the problem in equation (14) are:

$$\frac{u_c(c^y(x))}{u_c(e(s) - c^y(x))} = \frac{\beta}{\delta}\left(\frac{1 + \nu(\omega) + \eta(x)}{1 + \mu(x)}\right), \tag{15}$$

$$V_\omega(f(x)) = -\tfrac{\beta}{\delta}\left(\mu(x) - \zeta(x)\right) \tag{16}$$

where $\pi(s)\mu(x)$ are the multipliers associated with the participation constraints of the young (11), $\beta\pi(s)\zeta(x)$ is the multiplier on the upper bound for promised utility (12), $(\beta/\delta)\nu(\omega)$ is the multiplier associated with the promise keeping constraint (13) and $(\beta/\delta)\pi(s)\eta(x)$ are the multipliers associated with the non-negativity constraint on transfers. Given the concavity of the problem, these conditions are necessary and sufficient. There is also an envelope condition:

$$V_\omega(\omega) = -\tfrac{\beta}{\delta}\nu(\omega). \tag{17}$$

Equations (16) and (17) imply $\nu(\omega') = \mu(x) - \zeta(x)$.

The policy function $f(x)$, which is key to understanding the evolution of the intergen-

[19]The problem of finding $\omega_{\max}$ is straightforward because the value function $V(\omega)$ does not enter into the constraint set.

[20]Juan Pablo Rincón-Zapatero and Manuel S. Santos (2009) show how differentiability of value functions can be established quite generally when appropriate transversality conditions are met.

erational insurance rule, has the following properties.[21]

Lemma 2 *The policy function $f\colon \Omega \times \mathcal{S} \to \mathbb{R}$. (i) It is continuous and increasing in ω. $f(x) \geq \omega_0$ and strictly increasing in ω for $f(x) \in (\omega_0, \omega_{\max})$. (ii) There is at least one state r such that there is a $\omega^c(r) > \omega_0$ with $f(\omega, r) = \omega_0$ for $\omega \in [\omega_0, \omega^c(r)]$. (iii) For each state s there is a unique fixed point, $\omega^f(s)$, of the mapping $f(\omega, s)$. For at least one state $\omega^f(s) > \omega_0$. If $\omega^f(s) \in [\omega_0, \omega_{\max})$, then $c^y(\omega^f(s), s) = c^{y*}(s)$. If $\omega^f(s) = \omega_{\max}$, then $c^y(\omega_{\max}, s) \geq c^{y*}(s)$, with strict inequality if $\zeta(\omega_{\max}, s) > 0$. (iv) If the aggregate endowment is fixed, then $f(\omega, s)$ is decreasing in s and strictly decreasing for $f(\omega, s) \in (\omega_0, \omega_{\max})$.*

Figure 2 depicts an example of the policy functions $f(x)$ where there are three states and attention is restricted to $\omega \geq \omega_0$. The policy function $f(x)$ (likewise, the policy function $c^y(x)$) is continuous in ω simply because it is the solution to a strictly concave programming problem. The key properties of the policy function $f(x)$ are that for a given s, it is increasing in ω, cuts the 45° line once from above (see state 1 in Figure 2) and that there is some state such that $f(x)$ is flat at ω_0 for a range of $\omega > \omega_0$ (see states 2 and 3 in Figure 2).

It is intuitive that $f(x)$ is increasing in ω: a higher promise to the current old means lower consumption for the current young, and for states in which the participation constraint is binding, lower current consumption for the young requires a higher future promise, higher $f(x)$, to compensate. If the young participation constraint does not bind in state s, then it follows from the first-order conditions that

$$f(\omega, s) = \omega_0.$$

That is, the promised utility is *reset* to its initial level whenever the participation constraint does not bind. Not all participation constraints bind at ω_0 because, by Assumption 3, there are states where there is no transfer at ω_0. For these states the participation constraint of the young is strictly satisfied because $\omega_0 > \omega_{\min}$. This establishes property (*ii*) of Lemma 2 and provides a simple sufficient condition to prove the strong convergence result we establish in the next section.

To understand part (*iii*) of Lemma 2 suppose for simplicity that the non-negativity constraint on transfers and the upper bound constraint on future promises do not bind.

[21] To avoid the clumsy terminology of non-decreasing or weakly increasing, we describe a function h as increasing if $h(x) \geq h(y)$ whenever $x \geq y$. A function h is strictly increasing if $h(x) > h(y)$ whenever $x > y$.

That is, $\eta(x) = 0$ and $\zeta(x) = 0$ in equations (15) and (16). From equations (16) and (17) it can be seen that a fixed point of $f(x)$ requires $\mu(x) = \nu(\omega)$. Substituting this condition into (15) shows that the ratio of the marginal utilities equals β/δ, that is, consumption is at the first-best level: in particular, $c^y(\omega^f(s), s) = c^{y*}(s)$. Equally, for $f(x) > \omega$, where the promised expected utility is higher than today, it can be checked that the transfer from the young is below the first-best level; and for $f(x) < \omega$, where the promised expected utility is lower than today, the transfer from the young is above the first-best level. To show that $f(x)$ cuts the 45° degree line from above consider some $\omega > \omega^f(s)$ and suppose, to the contrary, that $f(x) \geq \omega$. This would imply that both the transfer from the young is lower at ω than $\omega^f(s)$, consumption of the young is higher, and the promised utility is higher. Since the participation constraint is binding at ω, the consumption and promise made at $\omega^f(s)$ cannot both be lower because this would violate the participation constraint. Hence, we can conclude that $f(x) < \omega$ for $\omega > \omega^f(s)$. A similar argument can be made to show that $f(x) > \omega$ for $\omega < \omega^f(s)$ and a detailed argument considering the non-negativity and upper bound constraints is given in the Appendix.

The policy functions need not be monotonic in the state. However, as shown in part (*iv*) of Lemma 2 they are ordered by the state if the aggregate endowment $e(s)$ does not depend on s (Figure 2 illustrates a case where the aggregate endowment is constant).[22] If the aggregate endowment is fixed, then $\lambda(s)$ is increasing in the state, by our convention on the ordering of states, and the endowment of the young, $e^y(s)$, is decreasing in s. In turn, this means that the autarky utility $\hat{v}(s)$ is also decreasing in s, because the future autarky utility is independent of the current state. Thus, where the participation constraint is binding in two states, the higher state will have lower consumption and lower promised utility for the young when compared to the lower state.[23] If the participation constraint does not bind, then $f(x) = \omega_0$ for any state. Hence, if the aggregate endowment is fixed, then $f(x)$ is weakly decreasing in s and strictly decreasing for $f(x) \in (\omega_0, \omega_{\max})$.[24]

For a fixed aggregate endowment, since the policy function $f(x)$ is monotone in the state, it also follows that $\omega^f(s) \geq \omega^f(r)$ for $e^y(s) > e^y(r)$, with strict inequality unless $\omega^f(s) = \omega^f(r) = \omega_0$ or $\omega^f(s) = \omega^f(r) = \omega_{\max}$.

[22] Whether the policy functions $f(x)$ are monotone or not in the state does not affect the conclusion about convergence that we discuss in the next section.

[23] In principle, the higher state could have a higher promised utility and a much lower consumption, but this will never be optimal.

[24] The result on monotonicity of the policy function $fx)$ in the state can be extended to the case where $e^y(1) \geq e^y(2) \geq \cdots \geq e^y(S)$ and $e(S) \geq e(S-1) \geq \cdots \geq e(1)$: that is young income weakly decreasing in the state but aggregate endowment weakly increasing in the state. Separately, it is also clear from the argument above that the monotonicity property will be preserved provided aggregate uncertainty is not too large.

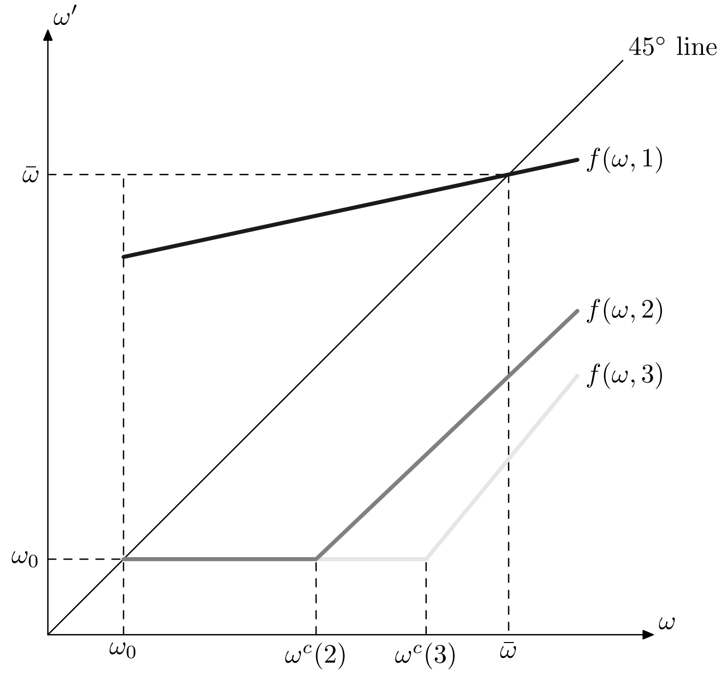

Figure 2: The Policy Functions $f(\omega, s)$.

Note: An illustration for $S = 3$. The functions $f(\omega, s)$ represent the future promises $\omega'(s)$ for s=1,2,3 as function of ω.

Figure 3 illustrates the evolution of promised utility for a given history of shocks corresponding to the three state example illustrated in Figure 2. The given history creates a particular sample path. Starting with ω_0, the sample path for the history $s^T = (s_0, s_1, \ldots, s_T)$ is constructed iteratively from the policy functions $f(\omega, s)$: $\omega_1 = f(\omega_0, s_0), \omega_2 = f(\omega_1, s_1), \ldots, \omega_{T+1} = f(\omega_T, s_T)$, where for each date $s_t \in \{1, 2, 3\}$. Figure 3 illustrates two important features. First, there is history dependence. That is, promised utility, and hence, consumption varies not only with the state but also on the past history of states. For example, at dates $t = 7$ and $t = 12$ state 1 occurs but the promised utility is different at the two dates. In particular, whenever state 1 occurs, the participation constraint of the young binds and a higher promised utility has to be offered to them so that they are willing to share more of their current relatively high endowment. Subsequent realizations of state 1 exacerbate the situation because the next generation of the young must also deliver on the past promises. This can be seen in Figure 3 with ω_t rising where the realization of state 1 is repeated. Secondly, Figure 3 shows that there are dates at which the promised utility is reset to ω_0. In the example, this happens whenever state 3 occurs and sometime when state 2 occurs. It can also be seen from the figure that whenever resetting occurs the subsequent sample path will be the same whenever the same sequence of states occurs. That is, the sample paths between resetting dates are probabilistically identical. This property allows us to establish convergence, which we discuss next.

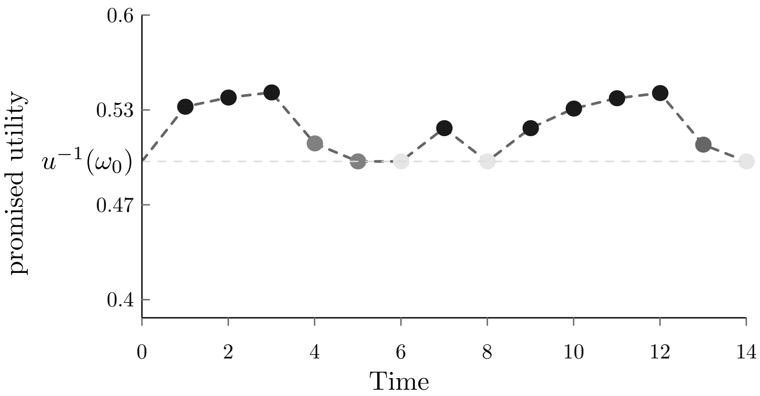

Figure 3: Sample Path of ω (converted to certainty equivalent consumption).
Note: The shade of the dots indicates the state s_t: dark gray for state 1, mid-gray for state 2 and light-gray for state 3.

5 Convergence to the Invariant Distribution

We now consider the long run distribution of the set of promised utilities. We show that there is strong convergence to a unique invariant distribution for any initial value of ω. Define $\bar{\omega} := \max_s\{\omega^f(s)\}$ to be the largest of fixed points of the policy function $f(\omega, s)$. It is possible that either $\bar{\omega} < \omega_{\max}$ or $\bar{\omega} = \omega_{\max}$. If $\bar{\omega} < \omega_{\max}$, then it is clear that any $\omega \in (\bar{\omega}, \omega_{\max}]$ is transitory and cannot be part of the invariant distribution. We show that the invariant distribution is non-degenerate with support in the interval $[\omega_0, \bar{\omega}]$. The convergence result relies on the monotonicity of the policy functions $f(\omega, s)$ in ω. Non-degeneracy of the distribution follows from Assumption 4 that the first-best is not sustainable.

Let the transition function for any ω and any set $A \subseteq [\omega_0, \bar{\omega}]$ be denoted by

$$P(\omega, A) = \Pr\{\omega_{t+1} \in A \mid \omega_t = \omega\} = \sum_s \pi(s)\mathbb{1}_A f(\omega, s),$$

where $\mathbb{1}_A f(\omega, s) = 1$ if $f(\omega, s) \in A$ and zero otherwise. The n-step transition function is

$$P^n(\omega, A) = \int P(\chi, A)P^{n-1}(\omega, \mathrm{d}\chi),$$

where $P^1(\omega, A) = P(\omega, A)$. An invariant distribution ϕ satisfies

$$\phi(A) = \int P(\omega, A)\phi(\mathrm{d}\omega), \quad \text{for all } A.$$

The initial distribution $\phi_0(A)$ at time $t = 0$ is the degenerate distribution with probability mass one on ω_0.

By part (*ii*) of Lemma 2 there is always some state such that $f(\omega, s) = \omega_0$ for $\omega \in (\omega_0, \omega^c(s))$. Therefore, there is some $N \geq 1$ and some $\epsilon > 0$ such that $P^N(\omega, \{\omega_0\}) > \epsilon$ for all $\omega \in [\omega_0, \bar{\omega}]$. This is obvious when the policy function is ordered by state: for any starting value of ω simply pick the highest state and consider the positive probability path where this state is repeated. Eventually ω_0 is reached. If the policy functions are not ordered by state, then a similar procedure is to take the sequence of states such that $f(\omega, s)$ is minimized at each stage along the path. This may be a sequence of different states but eventually there must be convergence with positive probability to reach ω_0. Standard results now apply. Since there is an $N \geq 1$ and $\epsilon > 0$ such that $P^N(\omega, \{\omega_0\}) > \epsilon$, it follows that Condition **M** of Nancy Stokey, Robert Lucas Jr. and Edward Prescott (p.348, 1989) is satisfied (see also, Jianjun Miao, 2014, Example 3.3.1, p.108). Then, Theorem 11.12 of Stokey, Lucas Jr. and Prescott (1989) shows strong convergence in the uniform metric to a unique probability measure $\phi(A)$ on $[\omega_0, \bar{\omega}]$. Moreover, $\phi(A)$ can be computed iteratively using the standard adjoint operator. We therefore obtain the following proposition.[25]

Proposition 4 *For any given initial distribution $\phi_0(A)$ for any $A \subseteq [\omega_0, \bar{\omega}]$, the sequence*

$$\phi_{t+1}(A) = \int P(\omega, A)\phi_t(\mathrm{d}\omega)$$

converges strongly to a unique invariant distribution $\phi(A)$, where $\phi(\{\omega_0\}) > 0$.

The central idea for this result is that there will always be a sequence of shocks such that eventually the participation constraint of the young does not bind. When this happens, the promise to that generation is *reset* to the minimum optimal level of promised utility ω_0. The process is regenerative, in the sense of Foss et al. (2018), and ω_0 is a regeneration point. Whenever the process reaches this regeneration point, the probabilistic future evolution of promises is the same independent of the time or past history that led to the regeneration point. This can be seen in Figure 3. There are regeneration points

[25] Ömer Açikgöz (2018), Sergey Foss, Vsevolod Shneer, Jonathan Thomas and Tim Worrall (2018) and Shenghao Zhu (2017) have used similar arguments to establish strong convergence in the case of a Bewley-Imrohoroglu-Huggett-Aiyagari precautionary savings model with heterogeneous agents.

at dates $t = 5$, $t = 6$, $t = 8$ and $t = 14$. The cycles between regeneration points are not identical but they are i.i.d. and therefore, eventually the distribution of future promises converges to the invariant distribution. In particular, the invariant distribution is independent of the initial distribution $\phi_0(A)$. Note that this property holds for any initial distribution $\phi_0(A)$ and not just the degenerate initial distribution we have assumed here. Note too that although eventually there is a stationary invariant distribution for the promises ω, for a given ω, at each date the promises for next period are determined by the policy function $f(\omega, s)$. Hence, unlike a situation where either transfers on the young can be enforced or there is no risk, a perpetual dynamics of consumption characterizes the optimal allocation even in the long run.

As a consequence of the properties of $f(\omega, s)$ given in Lemma 2, the invariant distribution ϕ is non-degenerate and has a mass point, or *atom*, at ω_0. Except in particular cases, it is not possible to give a more detailed characterization of the invariant distribution. However, in Section 7 we do compute the invariant distribution in a special case with just two possible endowment shocks and show that for some parameter values ϕ is a transformation of a geometric distribution with no mass at $\bar{\omega}$.[26]

A straightforward corollary of Proposition 4 is that there is a stationary invariant distribution for $x = (\omega, s)$. That is, for any $\mathcal{X} = A \times B$ where $A \subseteq [\omega_0, \bar{\omega}]$ and $B \subseteq \mathcal{S}$, we can define a transition function $P(x, \mathcal{X})$ and show strong convergence to an invariant distribution, say, $\varphi(\mathcal{X})$ where $\varphi(\mathcal{X}) = \phi(A)\pi(B)$ and $\pi(B) = \sum_{s \in B} \pi(s)$.[27] Given this invariant distribution, and the Markov property of the transition function, many of the results derived in finance theory to measure long term risk can be applied.

6 Measuring Generational Risk

We now turn to analyzing how risk is spread between generations across time. To this aim, we consider the pricing kernel implied by the solution for consumption characterized above (see, e.g., Gur Huberman, 1984; Gregory W. Huffman, 1986; Pamela Labadie, 1986). In our single agent, overlapping generations model, the pricing kernel for each $x_t = (\omega_t, s_t)$

[26] This result is not general and it is also possible to construct examples where $\bar{\omega} = \omega_{\max}$, for which the invariant distribution has a mass point at $\bar{\omega}$ for some parameter values and has no mass point at $\bar{\omega}$ for other parameter values.

[27] Where the transition function

$$P(x, \mathcal{X}) = \Pr\{(\omega', s') \in A \times B \mid x = (\omega, s)\} = \mathbb{1}_A f(\omega, s)\pi(B).$$

and $x_{t+1} = (f(\omega_t, s_t), s_{t+1})$ can be decomposed into three components:

$$m(x_t, x_{t+1}) = \delta \left(\frac{u_c(c^y(x_{t+1}))}{u_c(c^y(x_t))} \right) \left(\frac{\beta}{\delta} \frac{u_c(e(s_{t+1}) - c^y(x_{t+1}))}{u_c(c^y(x_{t+1}))} \right). \tag{18}$$

The three components are the social discount factor and the two bracketed terms representing the risk sharing across two adjacent generations and the risk sharing between generations at a given date.[28] If we would allow young to trade state contingent consumption claims, then the pricing kernel can be used to recursively determine the price of a k-period discount bond $p^k(x_t) = \sum_{s_{t+1}} q(x_t, x_{t+1}) p^{k-1}(x_{t+1})$ where $p^0(x_t) := 1$ and the state price $q(x_t, x_{t+1}) := \pi(s_{t+1}) m(x_t, x_{t+1})$. The associated continuously compounded yield is $y^k(x_t) := -(1/k) \log \big(p^k(x_t)\big)$.

A large difference between the true probability $\pi(s_{t+k})$ and the risk-adjusted probability $\varrho(x_t, x_{t+k}) := q(x_t, x_{t+k})/p^k(x_t)$ with $q(x_t, x_{t+k}) := \pi(s_{t+k}) \Pi_{j=1}^k m(x_{t+j-1}, x_{t+j})$ implies a large price of risk over the period t to $t+k$. Hence, the information embedded in the divergence between the two probabilities can be exploited to quantify the extent of risk borne by each generation in the constrained efficient allocation.

In the first-best allocation, if there is no aggregate risk, the value of the first bracketed term in equation (18) is unity and the second term is given by equation (7). If the non-negativity constraint on transfers does not bind, the second term is also unity and as a result $m(x_t, x_{t+1}) = \delta$ for each x_t and x_{t+1}. Hence, the yield curve is flat, that is, $y^k(x_t) = -\log(\delta)$ for each x_t and maturity k, so that pricing of bonds is risk neutral and the true and risk-adjusted probabilities coincide. This means that generations bear no risk.

In contrast, the first-order condition (15) implies that equation (18) is not constant over time. In particular, generations inheriting higher promise ω_t are less willing to substitute consumption when young for consumption when old, that is, $m(x_t, x_{t+1})$ decreases in ω_t. The pricing kernel also changes with s_t and s_{t+1}.[29] Hence, the true and risk-adjusted probabilities can no longer coincide, which implies that generations born in different states face different risk.

Following Backus, Chernov and Zin (2014), a convenient measure to quantify the divergence between the true and risk-adjusted probabilities over the period t to $t+k$ is

[28] Since shocks are transitory, there is no permanent component to the pricing kernel.

[29] If consumption can be ordered by state, e.g., if aggregate endowment is constant, then $m(x_t, x_{t+1})$ is decreasing in s_{t+1}. The dependence on s_t is not clear cut because a higher state s_t (low endowment of the young) means low consumption of the young but also a lower future promise, meaning lower old consumption at the next date. Thus, the impact on $m(x_t, x_{t+1})$ of s_t is ambiguous.

the conditional entropy $L^k(x_t) = -\mathbb{E}_t\left[\log\left(\varrho(x_t, x_{t+k})/\pi(s_{t+k})\right)\right]$.[30] Taking the average of $L^k(x_t)$ on the invariant distribution of x_t and normalizing it by the time horizon k, we define the mean entropy $I(k) := (1/k)\mathbb{E}\left[L^k(x_t)\right]$. The mean entropy is meaningful since it relates directly to the mean excess returns as $I(k) = -\mathbb{E}\left(\left|y^k(x_t) - y^\infty(x_t)\right|\right)$ where $y^\infty(x_t) := \lim_{k\to\infty} y^k(x_t)$ and captures the dynamics of yields' spreads and the associated risk to which each generation is exposed to by varying k. This is relevant here, since even temporary endowment shocks induce a persistent time variation in the expected consumption growth (Lemma 3). Let denote the matrix of state prices $q(x_t, x_{t+k})$ by Q and the corresponding eigenvectors and eigenvalue by $\psi(x_t)$ and ρ.[31] The following proposition states some important properties of generational risk associated with the entropy measure.[32]

Proposition 5 *(i) The yield on a very long-dated bond is $y^\infty(x_t) = -\log(\rho)$ for every x_t. (ii) The speed of convergence of the yield curve to the long yield is bounded by $\Upsilon := \log\left(\max_x \psi(x_t)/\min_x \psi(x_t)\right)$ as follows: $|y^k(x_t) - y^h(x_t)| \leq \left(\frac{1}{k} + \frac{1}{h}\right)\Upsilon$ for any x_t and $h, k > 0$ and $|y^k(x_t) - y^k(\hat{x}_t)| \leq \frac{2}{k}\Upsilon$ for any x_t and $\hat{x}_t$.*

Part (i) states that the yield of sufficiently long-dated bonds is invariant in x_t. This implies that the slope of the realized yield tends to decline over the time horizon. Hence, the persistent time variation in the autocorrelation of consumption between generations born over the period t to $t+k$ (which is an inverse measure of intergenerational consumption mobility) as induced by a temporary shock tends to vanish in the long run. The mean yield curve can lie above or below the long yield. If the mean yield curve is increasing (or $I(k)$ is decreasing), then short-run risk is smaller than long-run risk faced by generations. However, as we will show in the example economies, the opposite relation can also hold true. Since the mean yields' spread vanishes over the very long horizon, a syntethic measure of risk sharing is defined by the rate at which finite maturity yields converge to the long-run value. The metric Υ, which defines a bound on excess returns, serves this scope as stated in part (ii) of Proposition 5. If Υ is large, then the yield curves

[30] The insurance coefficient defined as $1 - cov[\Delta\log(c^y(x_t)), \Delta\log(e^y(s_t))]/var[\Delta\log(e^y(s_t))]$ could be used as an alternative measure to quantify the generational risk. However, the entropy measure has the advantage to be connected to excess returns on assets and real bond yields in a way that captures the dynamics of consumption.

[31] For simplicity, and because it mirrors our numerical procedure, we suppose that the invariant distribution of x_t is discrete and its sample space finite. This is not true in general and our convergence result in Proposition 4 does not restrict the limiting distribution. If the sample space is not finite, the pricing operator must be defined as $\mathbb{M}y(x_t) = \mathbb{E}[m(x_t, x_{t+1})y(x_{t+1}) \mid x_t]$ and instead of an eigenvector there is an eigenfunction $\psi(x_t)$ as a solution to the Perron-Frobenius eigenfunction problem: $\mathbb{M}\psi(x_t) = \rho\psi(x_t)$. See, Timothy M. Christensen (2017); Lars Peter Hansen and José A. Scheinkman (2009).

[32] Derivation of the entropy measure and the proof of Proposition 5 are reported in Supplementary Appendix B.

present large variations between different maturities as well as between different states fixing maturity. Hence, risk considerations are important. Notice that in the absence of aggregate risk, yield curves are flat and equal between maturities and states in the first best. Hence, the corresponding bound $\Upsilon^* = 0$.

We use the following section to illustrate the computation of equilibrium allocations for an illustrative set of examples, which will help to convey the qualitative properties of the equilibrium and to measure generational risk, so as embedded in the dynamics of the yield spreads under different parameter specifications.

7 Illustrative Examples

In this section, we study an economy where there are just two states $s = \{1, 2\}$, which realize with probability π and $1 - \pi$, respectively. We assume that the per-period utility function is of the CRRA form $u(c) = (c^{1-\gamma} - 1)/(1-\gamma)$, where γ is the risk aversion coefficient. Endowments are given in Table 1, where $\kappa > 1/2$ is the average endowment share of the young, θ is the variability of the aggregate endowment and $\sigma > 0$ is the variability of individual endowment shares.

Table 1: Endowments and Endowment Shares.

	$\frac{e^y(s)}{e(s)}$	$\frac{e^o(s)}{e(s)}$	$e(s)$
state 1:	$\kappa + \sigma$	$1 - \kappa - \sigma$	$1 - \theta\frac{1-\pi}{\pi}$
state 2:	$\kappa - \sigma\frac{\pi}{1-\pi}$	$1 - \kappa + \sigma\frac{\pi}{1-\pi}$	$1 + \theta$

An increase in θ is mean-preserving spread of the aggregate endowment. For $\theta = 0$, an increase in σ is mean-preserving spread of the endowment share. We use this parametric configuration to describe the optimal sustainable intergenerational insurance in three different example economies. In all examples, we ensure that Assumptions 1-3 are satisfied. Assumption 4 is violated in Example 1. In this case, we show that there is no consumption dynamics in the long run. Example 2 satisfies Assumption 4 but parameters are set so that young born in $s = 2$ are never constrained in the ergodic set of ω. In this case, we show that the long-run distribution of promised utilities is countable and geometric and derive an explicit formula for the risk measure Υ. In Example 3, the participation constraints of the young bind in both states for high level of ω and the feasibility constraint (12) eventually binds. In this case, the long-run distribution is negatively skewed and short-term bonds have higher yields than long-term bonds.

Example 1: Small Uncertainty

In this first example economy, Assumption 4 does not hold since first-best transfers do not violate the participation constraints of the young in any state. Uncertainty must be sufficiently low for this configuration to be an equilibrium outcome. A suitable set of parameters is $\gamma = 1$, $\theta = 0$, $\pi = 0.5$, $\kappa = 3/5$, $\sigma = 1/50$ and $\beta = \delta = \exp(-1/75)$. The value function and the policy functions are reported in Figure 5. For $\omega_0 = \omega^*$, there is full risk sharing and the promised utility remains constant over time. For ω_0 sufficiently larger than ω^*, the participation constraints of the young become eventually binding. The friction of limited enforcement generates partial risk sharing and a dynamics of promised utility with convergence to the stationary point ω^* within a finite time, similarly to the deterministic case of Section 3.

Example 2: Geometric Distribution

We now depart from the previous example by increasing income volatility to $\sigma = 1/10$. This specialization makes the participation constraint of the young violated in state 1 when transfers are first best. We also allow for aggregate uncertainty by setting $\theta = 1/10$. The value function and the policy functions are reported in Figure 6. In this example economy, $\bar{\omega} = \omega^f(1) < \omega_{\max}$. Hence, the upper bound constraint in equation (12) does not bind. Furthermore, the young are never constrained in state 2 for any ω belonging to the ergodic set. These properties considerably simplify the analysis. Indeed, future promise is reset to the level that maximizes the Pareto frontier and, in turn, the intergenerational risk-sharing possibilities as soon as the young are hit by an unfavorable income shock. That is, $f(\omega, 2) = \omega_0$ for $\omega \in [\omega_0, \bar{\omega}]$. The drop in future promises is drastic since the young are willing to participate in the risk-sharing arrangement for any level of past promises. In contrast, the young are always constrained if they experience a favorable income shock. To deter them from seceding, the optimal allocation requires back-loading of consumption: Future promise must be higher than the past one. In particular, future promise will only depend on the number of consecutive state 1s in the most recent history. For convenience of notation, denote by $\omega_0^{(n)}$ the utility promise after n consecutive state 1s starting from an initial promise of ω_0.[33] Since $\bar{\omega}$ is the fixed point of the policy function $f(\omega, 1)$, $\lim_{n\to\infty} \omega_0^{(n)} = \bar{\omega}$.[34]

It is easy to see that the invariant distribution is geometric on a countable sample space. Since the promised utility ω is reset to ω_0 every time that state 2 occurs, the probability that the promised utility is ω_0 is $1-\pi$, irrespective of the date or history. Therefore, T periods after regeneration, the distribution of ω is

$$\phi_T\left(\{\omega_0^{(n)}\}\right) = (1-\pi)\pi^n, \quad \text{for} \quad n = 0, 1, 2, 3, \dots, T-1, \quad \text{and} \quad \phi_T\left(\{\omega_0^{(T)}\}\right) = \pi^T.$$

The distribution ϕ_T satisfies the recursion

$$\phi_{T+1}\left(\{\omega_0)\}\right) = (1-\pi)\sum_{n=0}^{T} \phi_T\left(\{(\omega_0^{(n)}\}\right) \quad \text{and}$$
$$\phi_{T+1}\left(\{\omega_0^{(n+1)}\}\right) = \pi\phi_T\left(\{\omega_0^{(n)}\}\right) \quad \text{for} \quad n = 0, 1, 2, 3, \dots, T.$$

In the limit, ϕ_T converges to the invariant distribution $\phi(\{\omega_0^{(n)}\}) = (1-\pi)\pi^n$ for $n = 0, 1, \dots, \infty$, so as illustrated in Figure 4.

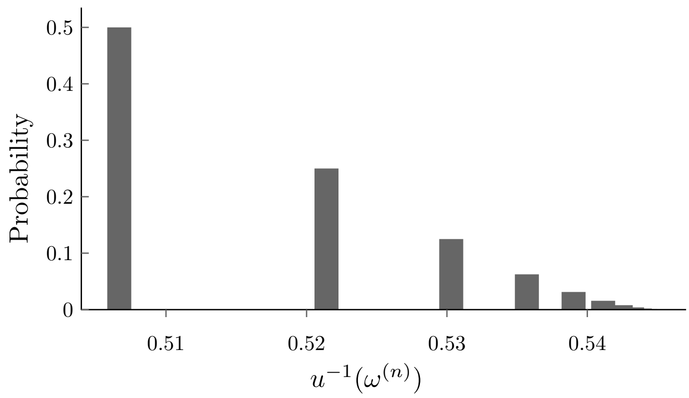

Figure 4: Probability mass function of ϕ (in terms of the certainty equivalent consumption).

Using the invariant distribution, it is immediate to compute the conditional expected

[33] Promised utility after the first state 1 is defined as $\omega_0^{(1)} := f(\omega_0, 1)$, after two consecutive state 1s as $\omega_0^{(2)} := f(f(\omega_0, 1), 1)$, and so on.

[34] In this setting, there are two alternative methods to calculate $\omega_0^{(n)}$. First, the value function and policy function can be derived by the value function iteration method described in the Section 4 where $\omega_0 = \max\{\omega \mid V_\omega(\omega) = 0\}$. Second, $\omega_0^{(1)}$ can be derived from the first-order conditions and the binding participation constraint in state 1. Since $\bar{\omega}$ is determined by the first-best consumption in state 1, and that $\lim_{n\to\infty} \omega_0^{(n)} = \bar{\omega}$, a forward shooting algorithm can be constructed to calculate ω_0. See the Supplementary Appendix for details of the shooting algorithm approach.

promised utility k periods forward from an initial period t:

$$\mathbb{E}\left[\omega_{t+k} \mid \omega_t = \omega_0^{(n)}\right] = \sum_{j=0}^{k-1}(1-\pi)\pi^j\omega_0^{(j)} + \pi^k\omega_0^{(n+k)}. \tag{19}$$

Since $\omega_0^{(k)}$ is monotone increasing in n, so too is the above conditional expectation. Thus, ω_t and $\mathbb{E}[\omega_{t+k} \mid \omega_t])$ are co-monotonic and the corresponding auto-covariance positive.[35] As $k \to \infty$, it can be seen from (19) that the conditional expectation converges to the mean of the invariant distribution, which is constant and independent of n. Hence, the absolute size of the auto-covariance tends to diminish with k because the occasional resetting means past history is forgotten and in the limit history is forgotten with probability one.[36]

We now use the equilibrium properties to quantify the generational risk. Panel (a) of Figure 8 shows the pricing kernels as functions of $\omega_0^{(n)}$ for each s and s'. All pricing kernels are monotonic decreasing in n: the longer the history of consecutive realizations of state $1s$, the higher the bond's yield which would induce generations, if they could, to save for their old age. The conditional entropy measure $L^k(\omega_0^{(n)}, s)$ directly follows. Panel (b) of Figure 8 shows how conditional entropy performs when $k = 1$, that is, it shows in which state $(\omega_0^{(n)}, s)$ the young bear the risk the most. For $s = 2$ the conditional entropy is invariant in n, since the risk borne by young born in the unfavorable state does not depend on the history of past shocks. This feature of the risk measure comes from the resetting property. On the other hand, for $s = 1$ the conditional entropy is increasing with n, since the young born in the favorable state and after a longer history of consecutive states 1s' realizations receive less insurance than those born after a shorter history. This property comes from the transmission into the future of uncertainty inherited from previous generations[37]

The yield curves implied by the pricing kernels, conditional on states $(\omega_0^{(n)}, s)$ for $n = \{0, \infty\}$ and $s = \{1, 2\}$, are shown in Panel A of Figure 9. All curves approach the

[35] This follows from Chebyshev's order inequality.

[36] The properties of the auto-covariance of promised utility are mirrored in the auto-covariance of individual consumption, as illustrated in Figure 7. Panel A shows that young consumption share is negatively correlated with individual endowment and that the conditional variance of consumption share declines for higher endowment values. Panel B reports the joint distribution of young consumption share of two adjacent generations, with consumption share of young born at time t on the x-axis and of young born at time $t+1$ on the y-axis. Conditional on young being born in state 2, the auto-covariance of consumption share is zero, while it is positive conditional on young being born in state 1. On average, the auto-covariance of consumption share is positive, that is, intergenerational consumption persistence arises.

[37] In the Supplementary Appendix B, we show that the conditional entropy and the insurance coefficient are cardinally equivalent risk measures under a geometric distribution of promised utility.

long-run yield $-\log(\rho)$, where $\rho = \delta$ since the non-negativity constraints of transfers and the feasibility constraint (12) do not bind. However, while some curves approach it from above, others do it from below, like, for example, when the state $(\omega_0^{(0)}, 2)$ occurs. Intuitively, if the current young is born in state 2 and past promise is ω_0, she obtains the highest insurance. From this initial period, the risk associated with future realizations of income shocks is spread into the future. Hence, long-run risk is higher than short-run risk, so as reflected in the upward sloping yield curve. In some other states, instead, yield curves are downward sloping since short-term risk is higher than long-term risk. As shown above, young are exposed to the highest risk if born in the favorable state after a long history of consecutive state 1. In this case, future realizations of bad income shocks for the young can lead to reset promised utility to ω_0 and, thus, to sustain higher level of insurance. Taking the average of yields on the invariant distribution of ω, we obtain an upward sloping mean yield curve (and therefore a downward sloping mean entropy $I(k)$, as show in Panel B of Figure 9) since, under the geometric distribution, the state $(\omega_0^{(0)}, 2)$ has the highest probability of realization. In the following proposition, we provide an analytical formulation of the risk measure Υ.

Proposition 6 *When $\phi(\{\omega_0^{(n)}\}) = (1-\pi)\pi^n$ for $n = 0, 1, \ldots, \infty$, the bound $\Upsilon = \Upsilon^* + \Upsilon^0$ where $\Upsilon^* = \log\left(\frac{1-\theta(1-\pi)/\pi}{1+\theta}\right)$ if $\theta \geq 0$ and $\Upsilon^* = \log\left(\frac{1+\theta}{1-\theta(1-\pi)/\pi}\right)$ if $\theta < 0$ and*

$$\Upsilon^0 = -\log\left(\frac{\delta}{\beta}\left(-1 + \left(\left(\tfrac{\beta+\delta}{\delta}\right)^{\frac{1+\beta\pi}{\beta(1-\pi)}} (\kappa+\sigma)^{\frac{1}{\beta(1-\pi)}} (1-\kappa-\sigma)^{\frac{\pi}{1-\pi}} \left(1-\kappa+\sigma\frac{1-\pi}{\pi}\right)\right)^{-1}\right)\right)$$

The reduction in risk sharing due to the friction of limited enforceability is measured by Υ^0. The larger Υ^0, the higher the yield curves' variability between maturities and states compared to the first-best case. Notice that changes of θ are fully absorbed in Υ^*. This implies that any shift in aggregate uncertainty does not alter the risk-sharing implications. From inspection of Υ it is immediate to see how risk sharing varies with other parameters. In particular, it decreases when discount factors decrease or when the individual income volatility increases.

Example 3: Non Geometric Distribution

Compared to Example 2, we now reduce the agents' willingness to participate to the insurance agreement by decreasing the level of discount factors to $\beta = \delta = 7/10$. In order to provide incentives and deter agents from deviating from the agreement, the difference in utility promises between the young born in state 1 and the young born in state 2 must be larger than in the case with a lower discount factor. Utility promises however cannot be too large. In particular they must satisfy the feasibility constraint (12). When many state 1s consecutively occur, utility promises approach the upper bound ω_{max} that mantains feasibility. At the same time, the participation constraint of the young born in the bad state starts binding when ω is high since the planner requires them to pay high transfers to keep past promises. To provide incentives the optimal allocation requires back-loading of consumption also when state 2 occurs. The value function and the equilibrium policies are depicted in Figure 10.

The long-run distribution is non geometric and negatively skewed. Resetting to ω_0 occurs gradually, while ω_{max} can realize with positive probability. The properties of the invariant distribution are reflected in the pricing kernels and implied conditional entropy (see, Figure 11). While state 1 is still the state in which the young bear the risk the most, young born in state 2 now receive less insurance for higher levels of past promises.

Yields curves approach the yield of a long-dated bond $-\log(\rho)$, which is now different from δ because the feasibility constraint (12) binds. Young are here exposed to the highest uninsured risk if born in state ω_{max} conditioned on any income shock. Hence, starting from ω_{max}, future realizations of bad income shocks for the young will lead to reset promised utility to ω_0 with a positive probability and, in turn, to provide more insurance in the long run than in the short run. Since state ω_{max} occurs with high probability, taking the average of yields on the invariant distribution results in a downward sloping mean yield curve (or upward sloping mean entropy), as illustrated in Figure 12. Unlike the case with a geometric invariant distribution, we cannot provide an analytical formula of the risk measure Υ. We can however determine it numerically. While in Example 2, $\Upsilon = 0.578$, here it is higher and equal to $\Upsilon = 0.7874$, meaning that less risk is shared.

8 Extensions

We have deliberately kept the model as simple as possible for reasons of readability and tractability. In particular, we have chosen an economic environment that is stationary to emphasize that the dynamics of the model derive from the participation constraints themselves and not from underlying features of the model. Nevertheless, our results and analysis are extendable in a number of directions, which we briefly discuss in the following sections.

Heterogeneous Preferences

The main model assumed that both young and old agents have the same preference function for consumption. This is inessential. Allowing for heterogeneity of preferences would complicate the notation but not change any of the results.[38]

As a special case, consider the situation in which the young have risk neutral preferences. In this case, the ordering of the states $\lambda(s) \geq \lambda(q)$ for $s > q$ implies that the endowment of the old is monotone in the state: $e^o(s) \geq e^o(q)$. It can then be checked that the policy function is also monotone in the state: $f(\omega, s) \leq f(\omega, q)$ for $s > q$. Likewise, consumption of the old is increasing in the state.[39]

Savings

We have assumed that the young do not have any access to a storage technology. This means the only method of insurance is through intergenerational transfers. Suppose that the young have access to a linear storage technology that delivers R units of endowment when old for every one unit stored when young. First, suppose that this access to storage is not available in autarky but only available to young agents who do not default on the transfers they are called on to make. It is clear that storage will not be used if its price $1/R$ is too high. That is, there is a $\bar{R}$ such that for $R < \bar{R}$, storage will not be used even if available. Given the solution to the optimal intergenerational insurance rule, $c^y(s^t)$, the upper bound is determined by

$$\bar{R}^{-1} = \max_{s^{t-1}} \left\{ \frac{\beta \mathbb{E}_{s_t}[u_c(e(s_t) - c^y(s^{t-1}, s_t))]}{u_c(c^y(s^{t-1}))} \right\}.$$

In the two state case, it is easy to check that the maximum occurs in state 1 when $\omega = \omega_0$. That is,

$$\bar{R}^{-1} = \frac{\beta \mathbb{E}_s[u_c(e(s) - c^y(f(\omega_0, 1), s))]}{u_c(c^y(\omega_0, 1))}.$$

[38] Sarolta Laczó (2015) has analyzed the implications of heterogeneous preference in a model with infinitely-lived agents.

[39] The participation constraint becomes $-\tau(s) + \beta\omega'(s) \geq \beta\omega_{\min}$. We have from the first-order condition that $\tau(\omega, \mu, e^o)$. A higher value of e^o will correspond to a lower value of μ for a given ω. Thus, $f(\omega, s)$ is weakly decreasing in s. Since the ratio of marginal utilities of the young and the old is decreasing in μ it is weakly increasing in e^o, and hence, the state. The consumption of the old is increasing in e^o, and hence, the state. The transfer is decreasing in e^o but not so much that consumption is offset. Hence, there is partial risk sharing.

In our example with $\gamma = 1.6$, we have $\bar{R} = 1.0065$. This shows that even when storage has a positive net return, the possibility of storage may have no impact on the optimal intergenerational insurance rule.

The situation is only slightly different if agents have access to the same storage technology in autarky. The autarky utility with storage a and gross rate of return R is

$$\hat{v}(r; R) = \max_{a \geq 0} u(e^y(r) - a) + \beta \sum_s \pi(s) u(e^o(s) + Ra).$$

Let $a(r)$ denote the solution. We have that $\hat{v}(r; 0)$ is the autarky value without storage and $\hat{v}(r; R)$ is increasing in R (the only dependence of this slope on $e^y(r)$ is through $e^o(r) = e(r) - e^y(r)$). The amount saved $a(r)$ is weakly increasing in R (strictly if $a(r) > 0$) and is one-to-one with $e^y(r)$ for $a(r) > 0$.

In this case, there will be a critical value $\bar{R}$ such that if $R < \bar{R}$, then the agents will not want to save. In our canonical example with logarithmic utility, the young agent will not wish to store at autarky when $R < (15/28)(1/\beta)$. If $\beta = 3/4$, then there is no saving if $R < 5/7$. We can show in this case that $\bar{R} < 1$. To show this, suppose $R = 1$. It is possible to compute $\hat{q}(1, 1) + \hat{q}(2, 2)$, the Perron root (from Section 2), as a function of β using the functions $\hat{v}(r, 1)$. It can be shown that this is an increasing function of β and so achieves a maximum of $(2 + \sqrt{129} - \sqrt{89})/4 < 1$ when $\beta = 1$. Since the Perron root is less than one, there are no transfers that can improve on autarky.[40]

A full analysis where the planner can use public storage is complicated because public storage adds a state variable in addition to the promised utilities. Árpád Ábrahám and Sarolta Laczó (2018) have solved this problem for two infinitely-lived agents. An analysis of the overlapping generations model with storage may be possible following their approach but is left for the subject of future research.

Growth

It has been assumed that the endowment is constant over time. This can be generalized to allow for growth in income along the lines set out in Fernando Alvarez and Urban J. Jermann (2001) and Ding Luo (2018) for infinitely-lived agents. Let $e^i_t(s)$ denote the endowment of agent i at date t with aggregate endowment $e_t(s) = e^y_t(s) + e^o_t(s)$. For simplicity, suppose that the aggregate endowment at each date is state independent and

[40] The same is true for any $R > 1$. If $R < 1$, then the Perron root may be greater than one. This occurs for example with $R = 13/14$ and $\beta = 21/26$.

that the growth factor $g = e_{t+1}/e_t$ is constant. Furthermore, suppose that the preference of both agents exhibit CRRA with risk aversion coefficient γ. Then, it is possible to rewrite the programming problem in a stationary way: normalize date t consumption by dividing by e_t, normalize date t utility variables, ω, V and $\hat{v}$, by dividing through by $e_t^{1-\gamma}$ and normalize the discount factors by multiplying by $g^{1-\gamma}$. The modified programming problem with these normalized variables is the same as the stationary problem set out in Section 4. Hence, the solution with constant growth is easily obtained by a simple reinterpretation of variables.[41]

Altruism

So far we have considered only purely selfish preferences without any role for altruism. To see how altruism can be incoporated into the model, we consider "warm glow" preferences (see, for example, James Andreoni, 1989) where the only departure from the setting described in Section 2 is that the old agent derives utility from leaving a bequest to the young agent. In this case, the intertemporal utility of an agent born after the history s^t is:

$$u\left(c^y\left(s^t\right)\right) + \beta \sum_{s_{t+1}} \pi\left(s_{t+1}\right)\left[u\left(c^o\left(s^{t+1}\right)\right) + \xi u\left(c^y\left(s^{t+1}\right)\right)\right],$$

where ξ captures the degree of preference for altruism:

$$c^y\left(s^t\right) = e^y\left(s_t\right) - \tau\left(s^t\right) + b\left(s^t\right) \text{ and } c^o\left(s^t\right) = e\left(s_t\right) - c^y\left(s_t\right),$$

and $b(s^t)$ is the bequest. Let $\hat{b}\left(s^t\right)$ denote the bequest from old to young in autarky.[42] The intertemporal utility of an agent born in state s in autarky is:

$$\hat{v}(s) = u\left(e^y(s) + \hat{b}(s)\right) + \beta \sum_r \pi(r)\left[u\left(e^o\left(r\right) - \hat{b}\left(r\right)\right) + \xi u\left(e^y\left(r\right) + \hat{b}\left(r\right)\right)\right].$$

The participation constraint for the old and the young are:

$$u\left(e\left(s\right) - c^y\left(s\right)\right) + \xi u\left(c^y\left(s\right)\right) \geq u\left(e^o\left(s\right) - \hat{b}\left(s\right)\right) + \xi u\left(e^y\left(s\right) + \hat{b}\left(s\right)\right)$$

and

$$u\left(c^y\left(s\right)\right) + \beta\omega'\left(s\right) \geq \hat{v}(s),$$

[41] Alvarez and Jermann (2001) show how variables can be reinterpreted if the growth rate is itself stochastic and Luo (2018) allows for heterogeneous preferences.

[42] If preferences exhibit CRRA with coefficient γ, then $\hat{b}(s_t) > 0$ for $\xi > (e^y(s_t)/e^o(s_t))^\gamma$, and zero otherwise.

and the promise keeping constraint is:

$$\sum_s \pi(s)\left(u\left(e(s) - c^y(s)\right) + \xi u\left(c^y(s)\right)\right) \geq \omega.$$

With these constraints, define Γ_ξ as the constraint set. Taking account of the altruistic preference weight that the old attach to the young, the value function for the social planner is rewritten as follows:

$$V(\omega) = \max_{\{c^y(s),\omega'(s)\}_s \in \Gamma_\xi} \left[\sum_s \pi(s)\left(\frac{\beta}{\delta} u\left(e(s) - c^y(s)\right) + \left(1 + \frac{\beta}{\delta}\xi\right) u\left(c^y(s)\right) + \delta V\left(\omega'(s)\right)\right)\right]$$

The first-order conditions for this problem are:

$$\frac{u_c\left(c^y(x)\right)}{u_c\left(e(s) - c^y(x)\right)} = \frac{\beta}{\delta}\left(\frac{1 + \nu(\omega) + \eta(x)}{1 + \mu(x) + \frac{\beta}{\delta}\xi\left(1 + \nu(\omega) + \eta(x)\right)}\right)$$

$$V_\omega\left(f(x)\right) = -\frac{\beta}{\delta}\left(\mu(x) - \zeta(x)\right)$$

where multipliers are specified as in Section 4, $x = (\omega, s)$ and $f(x)$ is the policy function for ω'. It is immediate to check that for $\xi = 0$, first-order conditions are those given by equations (15) and (16).

For a positive ξ makes $c^y(s)$ larger, $\mu(s)$ lower and, in turn, $\omega'(s)$ lower compared to the case without altruism. The dynamic properties of the intergenerational insurance rule remains however unaltered.

9 Discussion

It is worthwhile to contrast our results with the well-known results on risk sharing and limited commitment with infinitely-lived agents. The contrast can be explained most easily in the two-state case. Risk sharing with limited commitment in the case where there is two infinitely-lived agents with endowment risk has been considered by Thomas and Worrall (1988) and Kocherlakota (1996). Suppose there are just two states with one agent having higher income than the other in state 1 and the reverse true in state 2. In a partial insurance steady state, the ratio of marginal utilities of consumption are not completely equalized. There are two states corresponding to the transfer to be made by the relatively richer agent to the relatively poorer agent. There is convergence to this steady state. However, with just two states for the endowment process convergence to the steady-state occurs immediately both states have occurred. In the overlapping generations model considered here, even with two endowment states, if there is partial insurance, convergence occurs but not within finite time. Each generation faces only two potential ratios of marginal utilities, depending on the endowment state, but these two ratios may differ from one generation to the next depending on what was previously promised.

The steady-state solution of our model is more similar to the models with a continuum of infinitely-lived agents (see, e.g., Thomas and Worrall, 2007; Krueger and Perri, 2011; Broer, 2013). In that model there is a continuum of agents and each agent has high or low income with some fixed probability. In this case it is shown that if there is partial insurance, there is a finite set of possible transfers in steady state from the employed to the unemployed that depends on how many consecutive periods of unemployment the individual agent has experienced. This finite set is such that there are three different ratio of marginal utilities at any date, one each for constrained agents wether employed or unemployed and one for the unconstrained unemployed. Krueger and Perri (2011) and Broer (2013) do not address convergence to the steady state and although Thomas and Worrall (2007) characterize the solution out of steady state, they discuss convergence in only a particular special case. This is in contrast to the current paper where strong convergence is proved.

10 Conclusions

This paper has developed a theory of intergenerational insurance in a stochastic overlapping generations model where risk-sharing transfers must be voluntary. The model implies that (i) generational risk is partially spread over future generations in ways that create history dependence in transfers with periodic resetting, at which time the previous history is forgotten; (ii) risk sharing is partial, exhibiting heteroscedasticity in consumption with endowment and positive autocorrelation between the consumption of adjacent generations. The model can explain some features of observed consumption data left unexplained by other models.

We deliberately used a simple stationary environment with two-period overlapping generations to highlight the role played by risk and limited enforcement. It eschews some important aspects of intergenerational insurance. In particular, since there is a representative agent in each generation, there is no redistribution between agents of the same generation and since the single consumption good is non-storable, there is no saving.

The natural extension of this approach would then be to allow for a richer age structure involving more than two generations and a continuum of agents belonging to the same cohort, a process for endowments that incorporates a permanent as well as a transitory component, and to introduce a savings or storage technology. These tasks are left for future research.

References

Ábrahám, Árpád and Sarolta Laczó. 2018. "Efficient Risk Sharing with Limited Commitment and Storage." *The Review of Economic Studies*, 85(3): 1389–1424.

Açikgöz, Ömer. 2018. "On the Existence and Uniqueness of Stationary Equilibrium in Bewley Economies with Production." *Journal of Economic Theory*, 173: 18–25.

Aiyagari, S. Rao, and Dan Peled. 1991. "Dominant Root Characterization of Pareto Optimality and the Existence of Optimal Equilibria in Stochastic Overlapping Generations Models." *Journal of Economic Theory*, 54(1): 69–83.

Alvarez, Fernando, and Urban J. Jermann. 2001. "Quantitative Asset Pricing Implications of Endogenous Solvency Constraints." *Review of Financial Studies*, 14(4): 1117–1151.

Alvarez, Fernando, and Urban J. Jermann. 2005. "Using Asset Prices to Measure the Persistence of the Marginal Utility of Wealth." *Econometrica*, 73(6): 1977–2016.

Andreoni, James. 1989. "Giving with Impure Altruism: Applications to Charity and Ricardian Equivalence." *Journal of Political Economy*, 97(6): 1447–1458.

Azariadis, Costas, and Vincenzo Galasso. 2002. "Fiscal Constitutions." *Journal of Economic Theory*, 103(2): 255–281.

Backus, David, Mikhail Chernov, and Stanley Zin. 2014. "Sources of Entropy in Representative Agent Models." *The Journal of Finance*, 69(1): 51–99.

Ball, Laurence, and N. Gregory Mankiw. 2007. "Intergenerational Risk Sharing in the Spirit of Arrow, Debreu, and Rawls, with Applications to Social Security Design." *Journal of Political Economy*, 115(4): 523–547.

Bloise, Gaetano, and Filippo L. Calciano. 2008. "A Characterization of Inefficiency in Stochastic Overlapping Generations Economies." *Journal of Economic Theory*, 143(1): 442–468.

Borch, Karl. 1962. "Equilibrium in a Reinsurance Market." *Econometrica*, 30(3): 424–444.

Broer, Tobias. 2013. "The Wrong Shape of Insurance? What Cross-Sectional Distributions Tell Us about Models of Consumption Smoothing." *American Economic Journal: Macroeconomics*, 5(4): 107–140.

Chattopadhyay, Subir, and Piero Gottardi. 1999. "Stochastic OLG Models, Market Structure, and Optimality." *Journal of Economic Theory*, 89(1): 21–67.

Christensen, Timothy M. 2017. "Nonparametric Stochastic Discount Factor Decomposition." *Econometrica*, 85(5): 1501–1536.

Cooley, Thomas, and Jorge Soares. 1999. "A Positive Theory of Social Security Based on Reputation." *Journal of Political Economy*, 107(1): 135–160.

Demange, Gabrielle. 2009. "On Sustainable Pay-as-You-Go Contribution Rules." *Journal of Public Economic Theory*, 11(4): 493–527.

Demange, Gabrielle, and Guy Laroque. 1999. "Social Security and Demographic Shocks." *Econometrica*, 67(3): 527–542.

Diamond, Peter. 1977. "A Framework for Social Security Analysis." *Journal of Public Economics*, 8(3): 275–298.

Enders, Walter, and Harvey Lapan. 1982. "Social Security Taxation and Intergenerational Risk Sharing." *International Economic Review*, 23(3): 647–58.

Farhi, Emmanuel, and Iván Werning. 2007. "Inequality and Social Discounting." *Journal of Political Economy*, 115(3): 365–402.

Foss, Sergey, Vsevolod Shneer, Jonathan Thomas, and Tim Worrall. 2018. "Stochastic Stability of Monotone Economies in Regenerative Environments." *Journal of Economic Theory*, 173: 334–360.

Gordon, Roger, and Hal Varian. 1988. "Intergenerational Risk Sharing." *Journal of Public Economics*, 37(2): 185–202.

Hansen, Lars Peter, and José A. Scheinkman. 2009. "Long-Term Risk: An Operator Approach." *Econometrica*, 77(1): 177–234.

Huberman, Gur. 1984. "Capital Asset Pricing in an Overlapping Generations Model." *Journal of Economic Theory*, 33(2): 232–248.

Huffman, Gregory W. 1986. "The Representative Agent, Overlapping Generations, and Asset Pricing." *The Canadian Journal of Economics*, 19(3): 511–521.

Kocherlakota, Narayana 1996. "Implications of Efficient Risk Sharing without Commitment." *Review of Economic Studies*, 63(4): 595–610.

Krueger, Dirk, and Fabrizio Perri. 2011. "Public versus Private Risk Sharing." *Journal of Economic Theory*, 146(3): 920–956.

Krueger, Dirk, and Felix Kubler. 2006. "Pareto-Improving Social Security Reform when Financial Markets are Incomplete!?" *American Economic Review*, 96(3): 737–755.

Labadie, Pamela. 1986. "Comparative Dynamics and Risk Premia in an Overlapping Generations Model." *The Review of Economic Studies*, 53(1): 139–152.

Labadie, Pamela. 2004. "Aggregate Risk Sharing and Equivalent Financial Mechanisms in an Endowment Economy of Incomplete Participation." *Economic Theory*, 24(4): 789–809.

Laczó, Sarolta. 2015. "Risk Sharing with Limited Commitment and Preference Heterogeneity: Structural Estimation and Testing." *Journal of the European Economic Association*, 13(2): 265–292.

Ligon, Ethan, Jonathan P. Thomas, and Tim Worrall. 2002. "Informal Insurance Arrangements with Limited Commitment: Theory and Evidence from Village Economies." *Review of Economic Studies*, 69(1): 209–244.

Luo, Ding. 2018. "Heterogeneous Preferences and Asset Prices under Endogenously Incomplete Markets." Mimeo.

Manuelli, Rodolfo. 1990. "Existence and Optimality of Currency Equilibrium in Stochastic Overlapping Generations Models: The Pure Endowment Case." *Journal of Economic Theory*, 51(2): 268–294.

Martin, Ian, and Stephen Ross. 2019. "Notes on the Yield Curve." *Journal of Financial Economics.*

Merton, Robert C. 1983. "On the Role of Social Security as a Means for Efficient Risk Sharing in an Economy Where Human Capital Is Not Tradable." In: Zvi Bodie and John B. Shoven, eds., *Financial Aspects of the United States Pension System*, University of Chicago Press, 325–358.

Miao, Jianjun. 2014. *Economic Dynamics in Discrete Time.* Boston:MIT Press.

Prescott, Edward C., and José-Víctor Ríos-Rull. 2005. "On Equilibrium for Overlapping Generations Organizations." *International Economic Review*, 46(4): 1065–1080.

Rangel, Antonio. 2003. "Forward and Backward Intergenerational Goods: Why Is Social Security Good for the Environment?" *American Economic Review*, 93(3): 813–834.

Rangel, Antonio, and Richard Zeckhauser. 2000. "Can Market and Voting Institutions Generate Optimal Intergenerational Risk Sharing?" In: John Y. Campbell and Martin Feldstein, eds., *Risk Aspects of Investment-Based Social Security Reform, National Bureau of Economic Research Conference Report*, The University of Chicago Press, 113–152.

Rincón-Zapatero, Juan Pablo, and Manuel S. Santos. 2009. "Differentiability of the Value Function without Interiority Assumptions." *Journal of Economic Theory*, 144(5): 1948–1964.

Samuelson, Paul. 1958. "An Exact Consumption-Loan Model of Interest with or without the Social Contrivance of Money." *Journal of Political Economy*, 66(6): 467–482.

Shiller, Robert J. 1999. "Social Security and Institutions for Intergenerational, Intragenerational, and International Risk-sharing." *Carnegie-Rochester Conference Series on Public Policy*, 50: 165–204.

Stokey, Nancy, Robert Lucas Jr., with Ed Prescott. 1989. *Recursive Methods in Economic Dynamics.* Cambridge, Mass.:Harvard University Press.

Thomas, Jonathan, and Tim Worrall. 1988. "Self-Enforcing Wage Contracts." *Review of Economic Studies*, 55(4): 541–554.

Thomas, Jonathan, and Tim Worrall. 1994. "Foreign Direct Investment and the Risk of Expropriation." *Review of Economic Studies*, 61(1): 81–108.

Thomas, Jonathan, and Tim Worrall. 2007. "Unemployment Insurance under Moral Hazard and Limited Commitment: Public versus Private Provision." *Journal of Public Economic Theory*, 9(1): 151–181.

Zhu, Shenghao. 2019. "Existence of Equilibrium in an Incomplete-Market Model with Endogenous Labor Supply." *International Economic Review* , forthcoming.

Appendix

Proof of Lemma 1 We establish the domain and concavity and differentiability properties of the value function.

Domain: Since $\tau(s) \geq 0$ for all $s \in \mathcal{S}$, it is not feasible for $\omega < \omega_{\min}$ where $\omega_{\min}$ is the expected utility of the old in autarky. The largest feasible ω, that is, $\omega_{\max}$, can be found by solving the problem of choosing $(\tau(s), \omega'(s))$ to maximise $\sum_s \pi(s)u(e^o(s)+\tau(s))$ subject to $\tau(s) \geq 0$ and constraints (11) and (12). This is a strictly concave programming problem and the objective and constraint functions are continuous. Thus, there exists a unique solution. The constraint set is non-empty by Assumption 2. All constraints in (11) will bind at the solution: if one of these constraints did not bind, say in state r, then it would be possible to increase the maximand by increasing $\tau(r)$ without violating the other constraints. Equally, it is desirable to choose $\omega'(s)$ as large as possible because an increase in $\omega' s$ allows $\tau(s)$ to be increased without violating constraint (11), increasing the maximand. Thus, the solution involves $\omega'(s) = \omega_{\max}$ for each state s. Let $\tau^\sharp(s)$ denote the solution for the transfer and define $\omega^\sharp = \sum_s \pi(s)u(e^o(s)+\tau^\sharp(s))$. Since the constraint (11) binds in each state s:

$$\tau^\sharp(s) = e^y(s) - u^{-1}(u(e^y(s)) - \beta(\omega_{\max} - \omega_{\min})).$$

By definition $\omega_{\max} = \omega^\sharp$. Thus, $\omega_{\max}$ can be found as the root of

$$\sum_s \pi(s) \left(u(e(s) - u^{-1}(u(e^y(s)) - \beta(\omega - \omega_{\min}))) \right) - \omega.$$

We want to show that the root $\omega_{\max} \in (\omega_{\min}, \sum_s \pi(s)u(e(s)))$. This will follow if $\tau^\sharp(s) \in (0, e^y(s))$. We have $\omega_{\max} > \omega_{\min}$ by Assumption 2 and for any $\omega > \omega_{\min}$, $\tau^\sharp(s) > 0$. Now suppose $\tau^\sharp(r) = e^y(r)$ for some state r. Then, $u(e^y(r)) - \beta(\omega - \omega_{\min}) = u(0)$ or

$$u(0) \geq u(e^y(r)) - \beta \sum_s \pi(s) \left(u(e(s)) - u(e^o(s)) \right).$$

But this violates the Assumption 1. Hence, $\tau^\sharp(r) < e^y(r)$ for all r and consequently $\omega_{\max} < \sum_s \pi(s)u(e(s))$.

Concavity: We first show that $V(\omega)$ is concave. Consider the mapping T defined by

$$(TJ)(\omega) = \max_{\{c^y(s), \omega'(s)\} \in \Gamma} \left[\sum_s \pi(s) \left(\frac{\beta}{\delta} u(e - c^y(s)) + u(c^y(s)) \right) + \delta J(\omega'(s)) \right].$$

Consider $J = V^*$, the first-best frontier. It was established in Proposition 2 that $V^*(\omega)$ is concave. It follows from the definitions that $TV^*(\omega) \leq V^*(\omega)$ because $V^*(\omega) \leq v^*/(1-\delta)$ and because the maximization defined above adds the incentive constraints (11). Make the induction hypothesis that $T^n V^*(\omega) \leq T^{n-1} V^*(\omega)$ for $n \geq 2$ (we have just established that the inequality holds for $n = 1$). Applying the mapping T to both $T^n V^*(\omega)$ and $T^{n-1} V^*(\omega)$ shows that $T^{n+1} V^*(\omega) \leq T^n V^*(\omega)$ because the constraint set is the same in both cases but the objective is no greater in the former case, by the induction hypothesis. Hence, the sequence $T^n V^*(\omega)$ is non-increasing and converges. Let $V^\infty(\omega) = \lim_{n \to \infty} T^n V^*(\omega)$, the pointwise limit of the mapping T. We have that V^∞ and V are both fixed points of T. Since the mapping is monotonic, $T^n(V^*) \geq T^n(V) = V$. Hence $V^\infty \geq V$ but since V is the maximum, we have $V^\infty = V$.

Since $V^*(\omega)$ is concave and utility is concave, the objective function in the mapping T, starting from V^* is concave. The constraint set Γ is convex. Hence $TV^*(\omega)$ is concave. Hence, by induction, $T^n V^*(\omega)$ and the limit function V are concave.

Interiority: $\omega_0 \in (\omega_{\min}, \omega^*)$ Any non-trivial insurance makes some transfer to the old. Thus, by Assumption 2, $\omega_0 > \omega_{\min}$. It follows from the definition of ω_0 that $\nu(\omega_0) = 0$. It then follows from the first-order condition (15) that the transfer from the young satisfies $\tau(s) \leq \tau^*(s)$, with strict equality if the participation constraint of the young is non-binding ($\mu(\omega_0, s) = 0$). Thus, current expected utility of the old is not greater than ω^*. The inequality is strict by Assumption 4 that the first-best cannot be sustained.

Differentiability There are $2S$ choice variables and $3S + 1$ constraints including the non-negativity constraints on transfers. Without differentiability of the function V, the first-order condition (16) is replaced by

$$\partial V(f(\omega, s)) \ni -\tfrac{\beta}{\delta} \left(\mu(\omega, s) - \zeta(\omega, s) \right),$$

where $\partial V(\omega)$ denotes the set of superdifferentials of V at ω. Given the concavity of the function V, it is differentiable if the multipliers associated with the constraints are unique. The multipliers are unique if the linear independence constraint qualification is satisfied, that is, if the gradients of the binding constraints are linearly independent at the solution. It is easy to see that the participation constraint of the young and the old cannot simultaneously bind in a given state. Similarly for $\omega < \omega_{\max}$, it is easily checked that not all upper bound constraints can bind for all states. Thus, for $\omega < \omega_{\max}$, at most there can be $2S$ binding constraints. Since the marginal utilities are non-zero, $\beta > 0$, and $\pi(s) > 0$ for each s, it can be checked that the matrix of binding constraints has full rank. Hence, the multipliers are unique and $V(\omega)$ is differentiable on the interior of Ω. Since $V(\omega)$ is concave and differentiable, it is also continuously differentiable. It follows from the envelope condition (17) that $V_\omega(\omega_0) = 0$. Since the promise-keeping constraint (13) is an inequality, it is easily checked that $V(\omega)$ is non-increasing. The multiplier $\nu(\omega) = 0$ for $\omega < \omega_0$ and is increasing for $\omega > \omega_0$. Letting $\bar{\nu} := \lim_{\omega\to\omega_{\max}} \nu(\omega)$, where $\bar{\nu}$ may be infinite, we have $\lim_{\omega\to\omega_{\max}} V_\omega(\omega) = -\frac{\beta}{\delta}\bar{\nu}$, where $\bar{\nu} \in \mathbb{R}_+ \cup \{\infty\}$. □

Proof of Lemma 2 We first note that the old and young participation constraint do not bind simultaneously. If $\eta(\omega, s) > 0$, then young and old consume their endowments. By definition $\omega_0 = \max\{\omega \mid V_\omega(\omega) = 0\}$. For simplicity, define the following functions:

$$h(\omega) := -\tfrac{\delta}{\beta} V_\omega(\omega); \quad \rho^c(\omega) := \tfrac{\beta}{\delta}(1 + h(\omega)); \quad \upsilon(\omega, \mu; e) := u(\phi(\omega, \mu; e)) + \beta h^{-1}(\mu - \zeta);$$

where $\phi(\omega, \mu; e)$ and $\psi(\omega, e, \hat{\upsilon})$ are defined implicitly by:

$$\frac{\rho^c(\omega)}{1+\mu} = \frac{u_c(\phi(\omega, \mu; e))}{u_c(e - \phi(\omega, \mu; e))}; \quad \upsilon(\omega, \psi(\omega; e, \hat{\upsilon}); e) = \hat{\upsilon}. \quad \text{and} \quad \rho(\omega, \mu; e) := \frac{u_c(\phi(\omega, \mu; e))}{u_c(e - \phi(\omega, \mu; e))}.$$

The term $\phi(\omega, \mu; e)$ is the consumption of the young given ω, the multiplier μ and the aggregate endowment e. Likewise, $\upsilon(\omega, \mu; e)$ is the lifetime utility of the young agent and $\psi(\omega, e, \hat{\upsilon})$ is the value of the multiplier μ when the young participation constraint is binding. Here $h\colon \Omega \to [0, \bar{\nu}]$ and it is strictly increasing for $\omega > \omega_0$ with $h(\omega_{\max}) = \bar{\nu}$ and $h(\omega) = 0$ for $\omega \le \omega_0$. From (16), $\omega' = h^{-1}(\mu - \zeta)$. The function ϕ is continuous by the implicit function theorem because the derivative u_c and the function h are continuous. It can be checked that ϕ is increasing in aggregate endowment e (and $\partial\phi/\partial e < 1$), increasing in μ and decreasing in ω. Recall $\zeta > 0$ only if $\omega' = \omega_{\max}$ and hence, $\mu \ge \bar{\nu}$. For $\mu = 0$ (and hence, $\zeta = 0$), $h^{-1}(0) = \omega_0$, $\phi(\omega_0, 0; e) = c^{y*}(s)$, the first-best consumption of the young, and $\upsilon(\omega_0, 0; e) = u(c^{y*}(s)) + \beta\omega_0$. If follows from the properties of $\upsilon(\omega, \mu; e)$, that ψ is weakly increasing in ω (weakly because the solution may be $\mu = 0$), decreasing in e and increasing in $\hat{\upsilon}$.

(i) Since the constraint set is convex and the objective is strictly concave, the policy function $f(\omega, s)$ is single-valued and continuous in ω. It follows from above that

$\omega' = f(\omega, s) = \min\{h^{-1}(\psi(\omega; e, \hat{v})), \bar{\omega}\}$ and $f(\omega, s)$ is non-decreasing in ω. For $\psi = 0$, $f(\omega, s) = \omega_0$. For $\psi > 0$, $h^{-1}(\psi(\omega; e, \hat{v}))$ is strictly increasing in ω and hence, $f(\omega, s)$ is strictly increasing in ω provided $f(\omega, s) < \omega_{\max}$. If $\zeta(\omega, s) > 0$ ($h^{-1}(\psi(\omega; e, \hat{v}) > \omega_{\max}$), then $f(\omega, s) = \omega_{\max}$.

(ii) The critical threshold $\omega^c(e, \hat{v})$, above which the multiplier μ is positive, is determined by $\upsilon(\omega^c(e, \hat{v}), 0; e) = \hat{v}$. Thus, $\omega^c(e, \hat{v})$ is increasing in e and decreasing in $\hat{v}$. We want to show that there is some state r such that $\omega^c(e(r), \hat{v}(r)) > \omega_0$. Take state $r = S$. In state S, $\mu(\omega_0, S) = 0$. To see this, suppose to the contrary that $\mu(\omega_0, S) > 0$. Then $\eta(\omega_0, S) = 0$ and $\rho(\omega_0, \mu(\omega_0, S), e(S)) < \beta/\delta = \rho^c(\omega_0)$. Since there is no transfer from the old, $\phi(\omega_0, \mu(\omega_0, S), e(S)) \leq e^y(S)$ and hence, $\rho(\omega_0, \mu(\omega_0, S), e(S)) \geq \lambda(S)$. By Assumption 3, $\lambda(S) \geq \beta/\delta$, a contradiction. Hence, $\mu(\omega_0, S) = 0$. Since $\omega_0 > \omega_{\min}$, $\upsilon(\omega_0, 0; e(S)) = u(e^y(S)) + \beta\omega_0 > u(e^y(S)) + \beta\omega_{\min} = \hat{v}(S)$. Since $\upsilon(\omega, 0; e)$ is continuous and decreasing in ω, and since $\upsilon(\omega^c(e(S), \hat{v}(S)), 0; e(S)) = \hat{v}(S)$, it follows that $\omega^c(e(S), \hat{v}(S)) > \omega_0$, as required.

(iii) Existence of a fixed point $\omega^f(s)$ of the mapping $f(\omega, s)$ follows from the standard fixed point theorems given the continuity and monotonicity of $f(\omega, s)$ in ω. Part (ii) has shown that there is at least one state, namely state $s = S$, for which $\mu(\omega_0, s) = 0$. It must also be the case that there is another state r such that $\mu(\omega_0, r) > 0$. If this were not true, then since $\nu(\omega_0) = 0$, consumption is at the first best level in state r, that is, $c^y(\omega_0, r) = c^{y*}(s)$. If this were true for all states, then $\omega_0 = \omega^*$, the expected first-best utility of the old. But then Assumption 4 means that at least one of the constraints is strictly violated, and hence, there is a contradiction. Thus, there is at least one state $r \in \mathcal{S}$ such that $\mu(\omega_0, r) > 0$. We consider the two possibilities $\mu(\omega_0, s) = 0$ and $\mu(\omega_0, r) > 0$ in turn.

For states where $\mu(\omega_0, s) = 0$, then $\zeta(\omega_0, s) = 0$ and hence, $f(\omega_0, s) = h^{-1}(0) = \omega_0$, That is, ω_0 is a fixed point of the mapping $f(\omega, s)$. Since $f(\omega, s) \geq \omega_0$ (part (i)), there can be no fixed point $\omega^f(s) < \omega_0$.

Now consider a state where $\mu(\omega_0, s) > 0$. Since $\nu(\omega_0) = 0$ and $\eta(\omega_0, s) = 0$ by complementary slackness, it follows that $f(\omega_0, s) > \omega_0$ (with $f(\omega_0, s) = \omega_{\max} > \omega_0$ if $\zeta(\omega_0, s) > 0$). That is, any a state where $\mu(\omega_0, s) > 0$, fixed point $\omega^f(s) > \omega_0$. First, note that at a fixed point $\omega^f(s) > \omega_0$, $\mu(\omega^f(s), s) = \nu(\omega^f(s)) + \zeta(\omega^f(s), s)$. If $\zeta(\omega^f(s), s) > 0$, then $\nu(\omega^f(s)) = \bar{\nu}$ and $\omega^f(s) = \omega_{\max}$. This implies, from (15) that young consumption $c^y(\omega_{\max}, s) < c^{y*}(s)$ and $u(c^y(\omega_{\max}, s)) = u(e^y(s)) - \beta(\omega_{\max} - \omega_{\min})$. If $\zeta(\omega^f(s), s) = 0$, then $\mu(\omega^f(s), s) = \nu(\omega^f(s))$ and hence, from (15) $c^y(\omega^f(s), s) = c^{y*}(s)$. Hence, $\omega^f(s) = \omega_{\min} + \beta^{-1}(u(e^y(s)) - u(c^{y*}(s)))$. Taking each case together:

$$\omega^f(s) = \min\left\{\max\left\{\omega_0, \omega_{\min} + \beta^{-1}\left(u(e^y(s)) - u(c^{y*}(s))\right)\right\}, \omega_{\max}\right\}. \tag{20}$$

From Proposition 2, $c^{y*}(s)$ is unique, and hence, from equation (20) it follows that $\omega^f(s)$ is unique. There may, of course, be multiple states with the same fixed point.

(iv) Recall that $\omega'(\omega, e, \hat{v}) = \min\{h^{-1}(\psi(\omega, e, \hat{v})), \bar{\omega}\}$. It follows from the above that $\omega'(\omega, e, \hat{v})$ is decreasing in e and increasing in $\hat{v}$. To determine how $f(\omega, s)$ depends on s, we need to know how $e(s)$ and $\hat{v}(s)$ depend on s. When the aggregate endowment is fixed, it follows from our ordering of states that $e^y(1) \geq e^y(2) \geq \cdots \geq e^y(S)$. Hence, $\hat{v}(s)$ is decreasing in s. Moreover, for distinct states, s and r with $s < r$ and $e^y(s) > e^y(r)$, then $\hat{v}(s) > \hat{v}(r)$. Thus, $f(\omega, s) = \min\{h^{-1}(\psi(\omega, e, \hat{v}(s))), \bar{\omega}\}$ is decreasing in s. Moreover, for distinct states s and r such that $f(\omega, s) \in (\omega_0, \omega_{\max})$ and $f(\omega, r) \in (\omega_0, \omega_{\max})$, $f(\omega, s) > f(\omega, r)$ for $e^y(s) > e^y(r)$. We can order fixed points: $\omega^f(s) \geq \omega^f(r)$ for any state $s < r$, with strict inequality unless $\omega^f(s) = \omega^f(r) = \omega_0$ or $\omega^f(s) = \omega^f(r) = \omega_{\max}$.

□

Proof of Proposition 4 Since there is an $N \geq 1$ and $\epsilon > 0$ such that $P^N(\omega, \{\omega_0\}) > \epsilon$, it follows that Condition **M** of Stokey, Lucas Jr. and Prescott (p.348, 1989) is satisfied. Then, Theorem 11.12 of Stokey, Lucas Jr. and Prescott (1989) is used to establish strong convergence. □

Proof of Proposition 6 Let denote by $\nu^{(n)} := \nu(\omega_0^{(n)})$, $\mu^{(n)} := \mu(\omega_0^{(n)}, 1)$ and $\upsilon^{(n)} := 1 + \mu^{(n)}$ with the convention of $\upsilon^{(-1)} = 1$. The pricing kernel is denoted here by $m_{ss'}^{(n)} := m(x, x')$ with $x = (\omega_0^{(n)}, s)$ and $x' = (\omega_0^{(n+1)}, s')$ if $s = 1$ or $x' = (\omega_0^{(0)}, s')$ if $s = 2$, for each s' and n. When the non-negativity constraint of transfers does not bind, the equilibrium conditions in the case of a geometric long-run distribution, $\phi(\{\omega_0^{(n)}\}) = (1-\pi)\pi^n$ for $n = 0, 1, ...$, imply the following pricing kernels:

$$m_{11}^{(n)} = \delta\left(\frac{\frac{\beta}{\delta}(\upsilon^{(n)}) + \upsilon^{(n+1)}}{\frac{\beta}{\delta}(\upsilon^{(n-1)}) + \upsilon^{(n)}}\right); \quad m_{12}^{(n)} = \delta\left(\frac{e(1)}{e(2)}\right)\left(\frac{\frac{\beta}{\delta}\upsilon^{(n)} + 1}{\frac{\beta}{\delta}\upsilon^{(n-1)} + \upsilon^{(n)}}\right); \tag{21}$$

$$m_{21}^{(n)} = \delta\left(\frac{e(2)}{e(1)}\right)\left(\frac{\frac{\beta}{\delta} + \upsilon^{(0)}}{\frac{\beta}{\delta}\upsilon^{(n-1)} + 1}\right); \quad m_{22}^{(n)} = \delta\left(\frac{\frac{\beta}{\delta} + 1}{\frac{\beta}{\delta}\upsilon^{(n-1)} + 1}\right). \tag{22}$$

Taking the expectation of the pricing kernels over the stationary distribution, it can be checked that

$$\mathbb{E}\left[\log\left(m_{ss'}^{(n)}\right)\right] = \log(\delta).$$

Before determing the metric Υ in the constrained efficient allocation, we compute the same metric (denoted by Υ^*) in the first-best case ignoring the non-negativity constraints on transfers. In this case, $v^{(n)} = 1$ for any n and pricing kernels simplify to $m_{11}^{(n)} = \delta$, $m_{12}^{(n)} = \delta e(1)/e(2)$, $m_{21}^{(n)} = \delta e(2)/e(1)$ and $m_{22}^{(n)} = \delta$ for each n. We can therefore remove the superscript (n). The matrix of state prices is then equal to $Q^* = [\pi(s')\, m_{ss'}] \subset \mathbb{R}^{2\times 2}$. By The Perron–Frobenius theorem, there exists a unique (up to scale) eigenvector, ψ^*, and real eigenvalue ρ^*, satisfying $Q^*\psi^* = \rho^*\psi^*$, where $\psi^* = \{\psi_s^*\}_{s=1,2}$ and ρ^* are strictly positive and obtained by:

$$\begin{aligned}
\pi m_{11}\psi_1^* + (1-\pi)\, m_{12}\psi_2^* &= \rho^*\psi_1^* \\
\pi m_{21}\psi_1^* + (1-\pi)\, m_{22}\psi_2^* &= \rho^*\psi_2^*
\end{aligned}$$

Since the matrix Q^* is singular, solving the system we obtain $\rho^* = \pi m_{11} + (1-\pi)\, m_{22} = \delta$ and $\psi_1^* = e(1)/e(2)$ and $\psi_2^* = 1$. With aggregate endowments equal to $e(1) = 1 - \theta(1-\pi)/\pi$, $e(2) = 1+\theta$, we have $\psi_1^* < \psi_2^*$ for $\theta > 0$ and therefore $\Upsilon^* = \log\left(\frac{\max_s \psi_s^*}{\min_s \psi_s^*}\right) = \log\left(\frac{1+\theta}{1-\theta(1-\pi)/\pi}\right)$. In particular, when $\pi = 1/2$, $\Upsilon^* = \log\left(\frac{1+\theta}{1-\theta}\right)$.[43]

Using the equilibrium pricing kernels above we can now determine the bound Υ. The eigenvector ψ and eigenvalue ρ satisfying $Q\psi = \rho\psi$ are obtained solving the system:

$$\pi m_{11}^{(n)}\psi_1^{(n+1)} + (1-\pi) m_{12}^{(n)}\psi_2^{(n+1)} = \rho\psi_1^{(n)} \quad n = 0,1,\ldots, \tag{23}$$

$$\pi m_{21}^{(n)}\psi_1^{(0)} + (1-\pi) m_{22}^{(n)}\psi_2^{(0)} = \rho\psi_2^{(n)} \quad n = 0,1,\ldots. \tag{24}$$

Using the system of equations (23) and (24), we construct the eigenvector $\psi^* = \left\{\psi_s^{(n)}\right\}_{s=1,2}^{n=0,1,2,\ldots}$ corresponding to the eigenvalue $\rho = \delta$. Since $e(2) > e(1)$ for $\theta > 0$ and $m_{21}^{(0)} > \delta > m_{12}^{(\infty)}$, then[44]

$$\psi_{\max} := \left\{\psi_s^{(n)}\right\}_{s=1,2}^{n=0,1,2,\ldots} = \psi_2^{(0)} = \frac{m_{21}^{(0)}}{\delta}\psi_1^{(0)}$$

$$\psi_{\min} := \left\{\psi_s^{(n)}\right\}_{s=1,2}^{n=0,1,2,\ldots} = \psi_1^{(\infty)} = \frac{m_{12}^{(\infty)}}{\delta^2}\left(\pi m_{21}^{(\infty)} + (1-\pi) m_{22}^{(\infty)}\frac{m_{21}^{(0)}}{\delta}\right)\psi_1^{(0)}.$$

[43] For $\theta < 0$, $\Upsilon^* = \log e(1) - \log e(2)$.

[44] For $\theta < 0$ provided that θ is not too small, the ordering of ψ_1 and ψ_2 may be reversed.

Hence,

$$\frac{\psi_{\min}}{\psi_{\max}} = \frac{m_{12}^{(\infty)}}{\delta}\left(\pi\frac{m_{21}^{(\infty)}}{m_{21}^{(0)}} + (1-\pi)\frac{m_{22}^{(\infty)}}{m_{22}^{(0)}}\right) = \frac{e(1)}{e(2)}\frac{1}{\upsilon^{(\infty)}}.$$

which gives

$$\Upsilon := \log\left(\frac{\psi_{\max}}{\psi_{\min}}\right) = \log\left(\frac{e(2)}{e(1)}\right) + \log(\upsilon^{(\infty)}).$$

Let $\Upsilon^0 := \log(\upsilon^{(\infty)})$ denote the value when there is no aggregate uncertainty ($\theta = 0$). Then $\Upsilon = \Upsilon^* + \Upsilon^0$. Replacing the value of $\upsilon^{(\infty)}$ (which is determined in the section "Shooting Algorithm" in the Supplementary Appendix), we obtain

$$\Upsilon^0 = -\log\left(\frac{\delta}{\beta}\left(-1 + \left(\left(\tfrac{\beta+\delta}{\delta}\right)^{\frac{1}{\beta(1-\pi)}}\left(\tfrac{\beta+\delta}{\beta}\right)^{\frac{\pi}{1-\pi}}\left(\tfrac{e^y(1)}{e(1)}\right)^{\frac{1}{\beta(1-\pi)}}\left(\tfrac{e^o(1)}{e(1)}\right)^{\frac{\pi}{1-\pi}}\tfrac{e^o(2)}{e(2)}\right)^{-1}\right)\right).$$

$\square$

Figures

This Appendix presents the figures, which we refer to in the text.

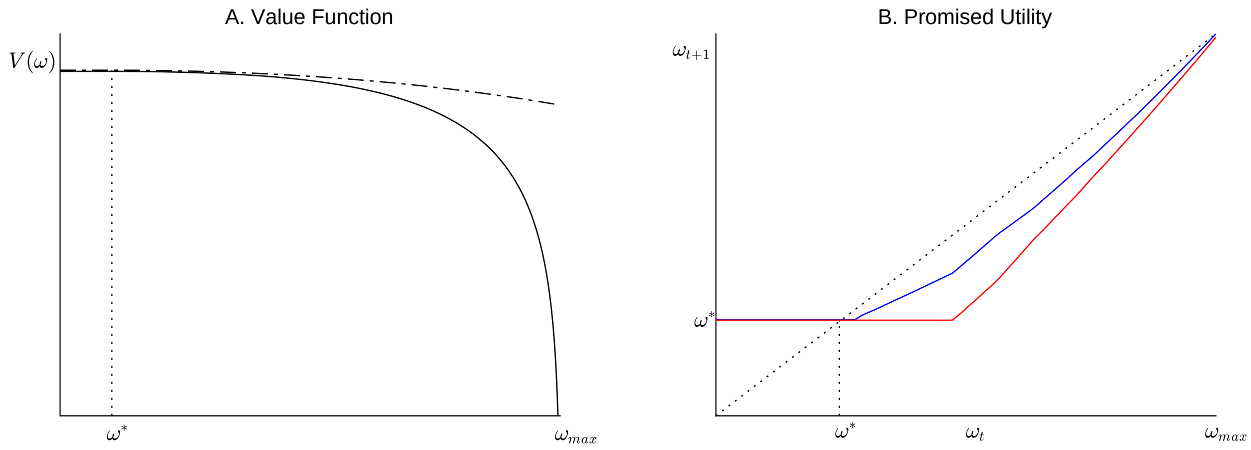

Figure 5: Value Function and Equilibrium Policies - Example 1.

Note: In Panel A, the solid line is the value function in the constrained efficient allocation and the dashed line is the value function in the first-best case. In Panel B, the dotted line is the 45 degree line, the blue line is the policy $\omega'(1)$ and the red line is the policy $\omega'(2)$.

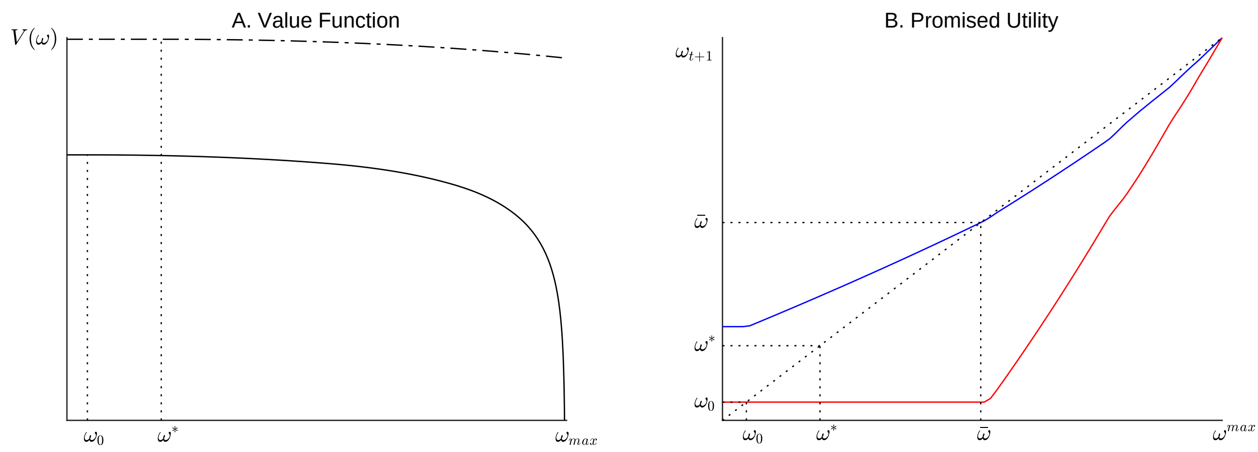

Figure 6: Value Function and Equilibrium Policies - Example 2.

Note: In Panel A, the solid line is the value function in the constrained efficient allocation and the dashed line is the value function in the first-best case. In Panel B, the dotted line is the 45 degree line, the blue line is the policy $\omega'(1)$ and the red line is the policy $\omega'(2)$.

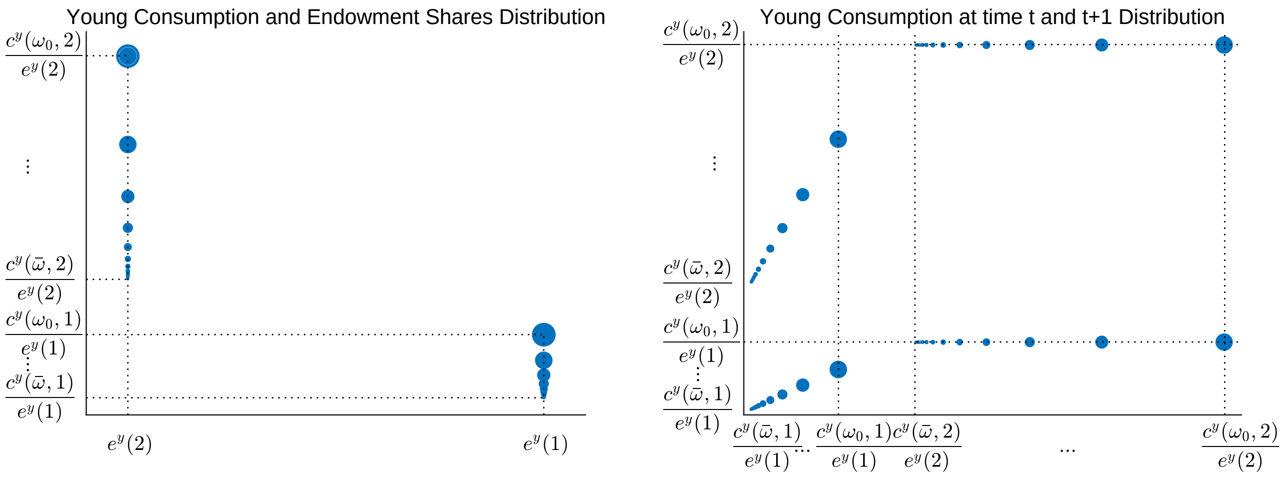

Figure 7: Auto-correlation of Young Consumption - Example 2.

Note: In Panel A, each dot represents the frequency of the occurrence $(e^y(s), c^y(x))$ in the long-run distribution. In Panel B, each dot represents the frequency of the occurrence $(c^y(x)), c^y(x'))$ in the long-run distribution. In both panels, the bigger the dot is, the higher is the frequency.

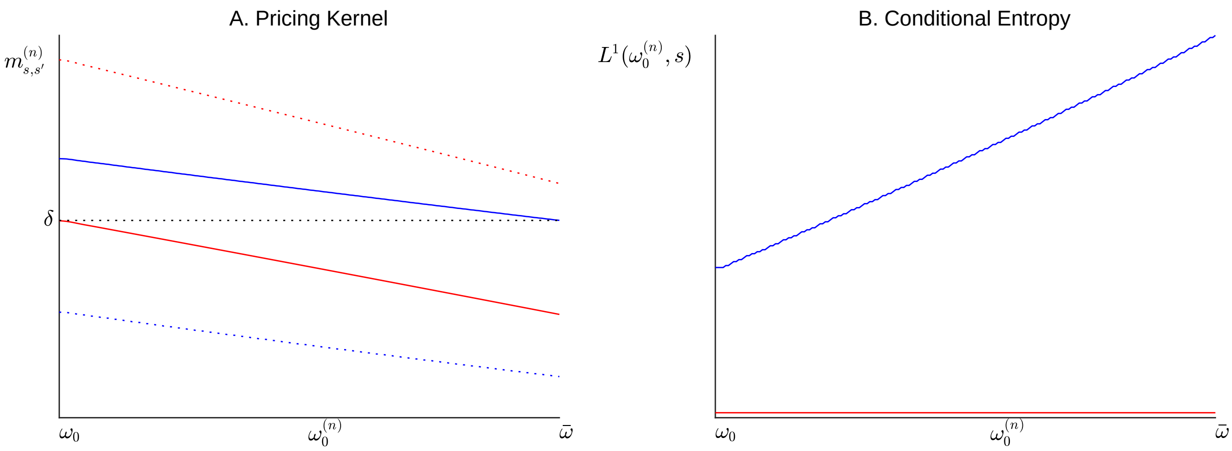

Figure 8: Pricing Kernel and Conditional Entropy - Example 2.

Note: In Panel A, the blue solid line is $m_{11}^{(n)}$, the blue dashed line is $m_{12}^{(n)}$, the red solid line is $m_{22}^{(n)}$, and the red dashed line is $m_{21}^{(n)}$. In Panel B, the blue line is $L^1(\omega_0^{(n)}, 1)$ and the red line is $L^1(\omega_0^{(n)}, 2)$.

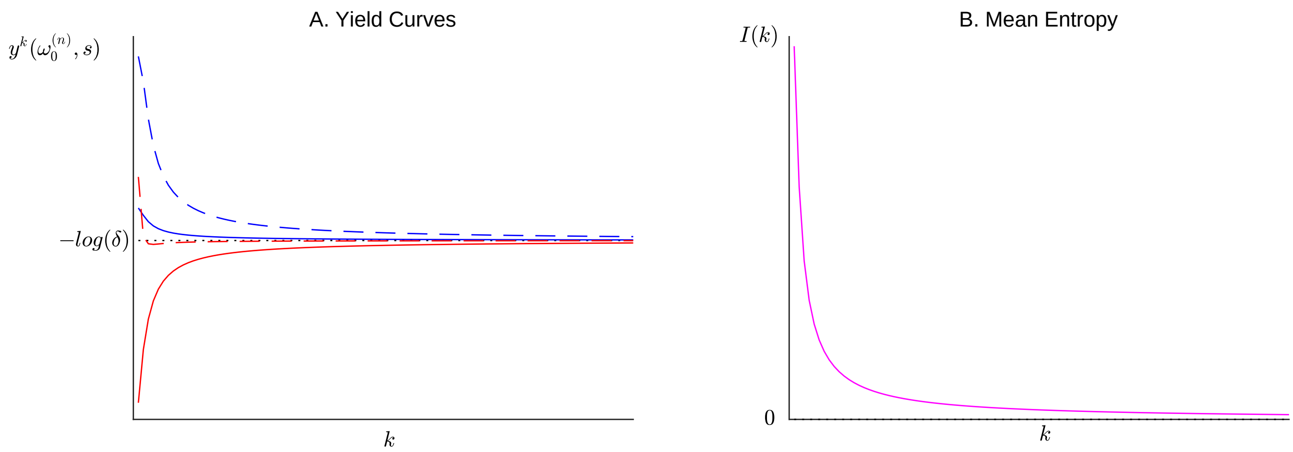

Figure 9: Yield Curves and Mean Entropy - Example 2.

Note: In Panel A, the blue solid line is $y^k(\omega_0, 1)$, the blue dashed line is $y^k(\bar{\omega}, 1)$, the red solid line is $y^k(\omega_0, 2)$, the red dashed line is $y^k(\bar{\omega}, 2)$. In Panel B, the blue line is $L^1(\omega_0^{(n)}, 1)$ and the red line is $L^1(\omega_0^{(n)}, 2)$.

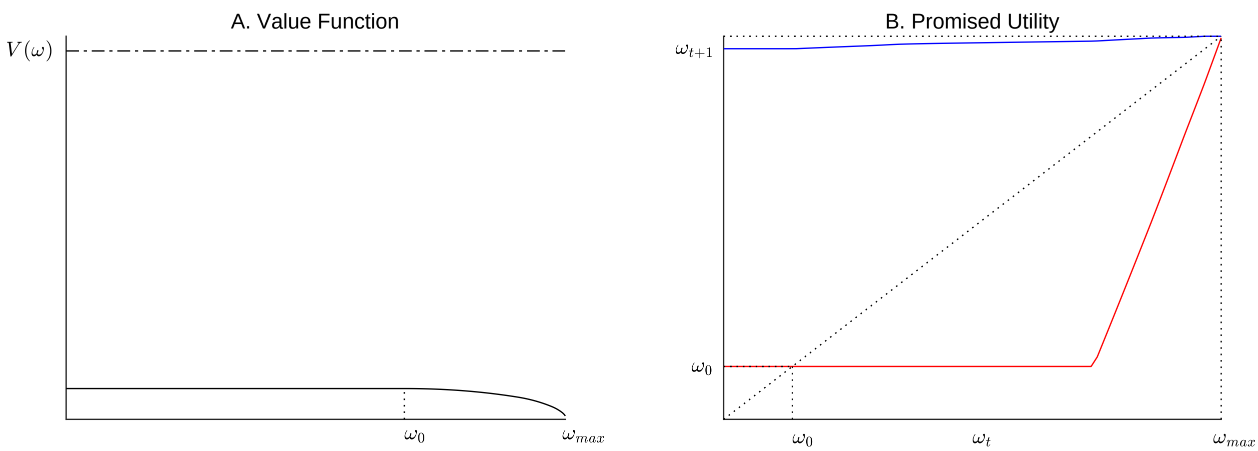

Figure 10: Value Function and Equilibrium Policies - Example 3.

Note: In Panel A, the solid line is the value function in the constrained efficient allocation and the dashed line is the value function in the first-best case. In Panel B, the dotted line is the 45 degree line, the blue line is the policy $\omega'(1)$ and the red line is the policy $\omega'(2)$.

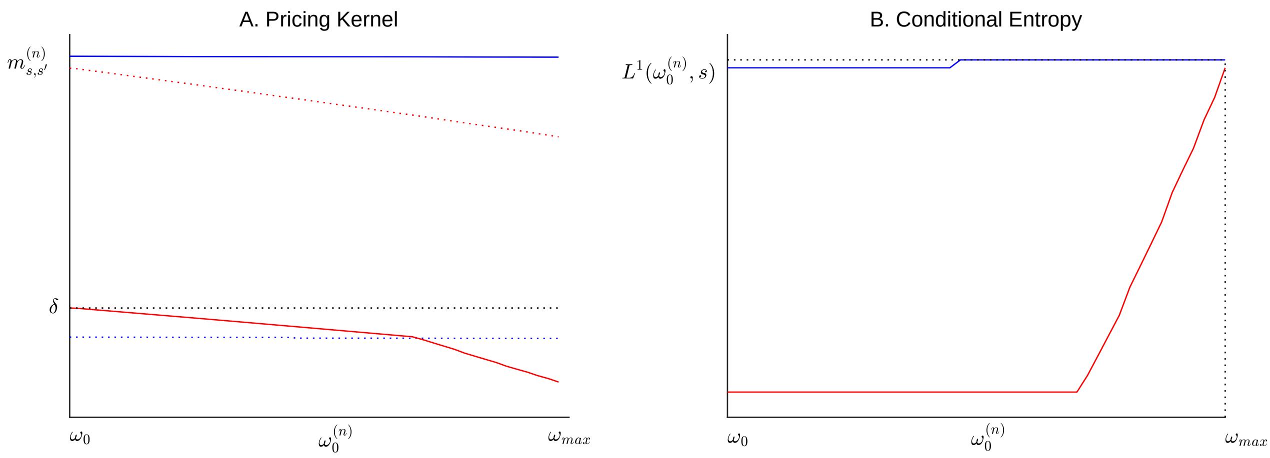

Figure 11: Pricing Kernel and Conditional Entropy - Example 3.

Note: In Panel A, the blue solid line is $m^{(n)}_{11}$, the blue dashed line is $m^{(n)}_{12}$, the red solid line is $m^{(n)}_{22}$, and the red dashed line is $m^{(n)}_{21}$. In Panel B, the blue line is $L^1(\omega_0^{(n)}, 1)$ and the red line is $L^1(\omega_0^{(n)}, 2)$.

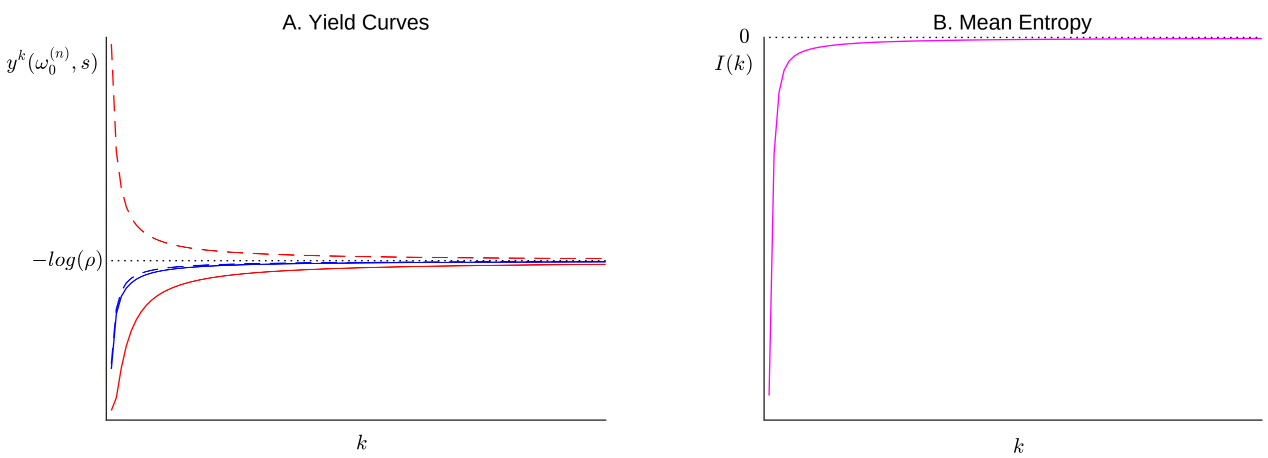

Figure 12: Yield Curves and Mean Entropy - Example 3.

Note: In Panel A, the blue solid line is $y^k(\omega_0, 1)$, the blue dashed line is $y^k(\bar{\omega}, 1)$, the red solid line is $y^k(\omega_0, 2)$, the red dashed line is $y^k(\bar{\omega}, 2)$. In Panel B, the blue line is $L^1(\omega_0^{(n)}, 1)$ and the red line is $L^1(\omega_0^{(n)}, 2)$.

Supplementary Appendix

These appendices present supplementary material referenced in the paper. Appendix A provides proofs of Propositions 1 and 3. Appendix B derives risk measures and provides the proof of Proposition 5. Appendix C presents the shooting algorithm used to derive the optimal allocation of Example 2 in Section 7. Appendix D describes the psedo-code for the numerical algorithm.

Appendix A: Proof of Propositions 1 and 3

Proof of Proposition 1 The autarky utility of all agents in state r is given by:

$$\hat{v}(r) := u(e^y(r)) + \beta \sum_s \pi(s) u(e^o(s)).$$

Consider a small transfer of $\mathrm{d}\tau(s)$ in state s from the young to the old. The problem of existence of a sustainable allocation can be answered by finding a vector of positive transfer $\mathrm{d}\tau$ such that the autarky utility of agents is not decreased in any state and increased in at least one state. Differentiating, the change in the autarky utility is non-negative if

$$-u_c(e^y(r))\mathrm{d}\tau(r) + \beta \sum_s \pi(s) u_c(e^o(s))\mathrm{d}\tau(s) \geq 0. \tag{25}$$

Rearranging (25) in terms of the marginal rates of substitution $\hat{m}(r, s)$ we have

$$-\mathrm{d}\tau(r) + \sum_s \pi(s)\hat{m}(r, s)\mathrm{d}\tau(s) \geq 0.$$

With $\hat{q}(r, s) = \pi(s)\hat{m}(r, s)$, and writing in matrix notation, the problem of existence can then be addressed by finding a vector $d\tau > 0$ that solves

$$\left(\hat{Q} - I\right) \mathrm{d}\tau \geq 0 \tag{26}$$

where I is the identity matrix and $\hat{Q}$ is the matrix of $\hat{q}(r, s)$. Equation (26) has a well-known solution. Using the Perron-Frobenius theorem, that there exists a strictly positive solution for $\mathrm{d}\tau$, provided the Perron root, that is the largest eigenvalue, of $\hat{Q}$ is greater than one. A lower bound for the Perron root is given by the minimum row sum of $\hat{Q}$ (the Frobenius lower bound). Thus, Assumption 2 is sufficient for the Perron root to be greater than one and hence the existence of positive transfers from the young to the old that improve the utility of each generation. Hence, there exists a non-trivial sustainable allocation under assumption 2. $\square$

Proof of Proposition 3 As explained in the text, with $\hat{v} = u(e^y) + \beta u(e^o)$, there is a τ_1^c defined by

$$u(e^y - \tau_1^c) + \beta u(e^o + \tau^*) = \hat{v}.$$

Define τ_j^c recursively by

$$u(e^y - \tau_j^c) + \beta u(e^o + \tau_{j-1}^c) = \hat{v} \quad \text{for } j = 1, 2, \ldots.$$

From the strict concavity of the utility function, $\tau_j^c > \tau_{j-1}^c$, and $\lim_{j\to\infty} \tau_j^c = \bar{\tau} = e^y - e^o$. Correspondingly, define $\omega_j^c = u(e^o + \tau_j^c)$. We have $\omega_0^c = \omega^*$ and $\lim_{j\to\infty} \omega_j^c = \bar{\omega}$. Let $v^* = u(e^y + \tau^*) + \frac{\beta}{\delta}\omega^*$. Write $V_n(\omega)$ for the value function for $\omega \in (\omega_{n-1}^c, \omega_n^c]$. We have

$$V_n(\omega) = u(e - u^{-1}(\omega)) + \tfrac{\beta}{\delta}\omega + \delta V_{n-1}\left(\tfrac{1}{\beta}\left(\hat{v} - u(e - u^{-1}(\omega))\right)\right).$$

For $\omega \leq \omega^*$, we have $\tau(\omega) = \tau^*$ and $\omega' = \omega^*$. Therefore, $V(\omega) = v^*/(1-\delta)$ and $V(\omega)$ is constant on the interval $u(e^o), \omega^*]$. For $\omega \in (\omega^*, \omega_1^c]$

$$V_1(\omega) = u(e - u^{-1}(\omega)) + \tfrac{\beta}{\delta}\omega + \frac{\delta}{1-\delta}v^*.$$

The function $V_1(\omega)$ is differentiable because u is strictly increasing and differentiable. Denoting the derivative with a $\prime$, differentiating $V_1(\omega)$ gives

$$V_1'(\omega) = \frac{\beta}{\delta} - \frac{u_c(e - u^{-1}(\omega))}{u_c(u^{-1}(\omega))}.$$

Let $\rho(\omega) := u_c(e - u^{-1}(\omega))/u_c(u^{-1}(\omega))$. Since $\omega > \omega^*$, $\rho(\omega) > \frac{\beta}{\delta}$. Thus, $V_1'(\omega) < 0$. Note that $\rho(\omega^*) = \frac{\beta}{\delta}$ and therefore in the limit as $\omega \downarrow \omega^*$, $V_1'(\omega) = 0$. Also since $\rho(\omega)$ is increasing (because of strict concavity of u), the function $V_1(\omega)$ is strictly concave. Having established that $V_1(\omega)$ is decreasing and strictly concave, we can proceed by induction and assume $V_{n-1}(\omega)$ is decreasing and strictly concave to straightforwardly establish that $V_n(\omega)$ is decreasing and strictly concave. It is easy to establish continuity by showing that $\lim_{\omega\downarrow\omega_n^c} V_{n+1}(\omega) = V_n(\omega_n^c)$. We want to establish differentiability too by showing that

$$\lim_{\omega\downarrow\omega_n^c} V_{n+1}'(\omega) = V_n'(\omega_n^c).$$

For $\omega \in (\omega_n^c, \omega_{n+1}^c)$ we have

$$V_{n+1}'(\omega) = \frac{\beta}{\delta} - \rho(\omega)\left(1 - \frac{\delta}{\beta} V_n'(\omega')\right).$$

Starting with $n = 1$, we have

$$\lim_{\omega \downarrow \omega_1^c} V_2'(\omega) = \frac{\beta}{\delta} - \rho(\omega_1^c)\left(1 - \delta \lim_{\omega \downarrow \omega_0^c} V_1'(\omega)\right).$$

Since $\lim_{\omega \downarrow \omega_0^c} V_1'(\omega) = 0$, we have

$$\lim_{\omega \downarrow \omega_1^c} V_2'(\omega) = \frac{\beta}{\delta} - \rho(\omega_1^c) = V_1'(\omega_1^c).$$

Therefore, make the recursive assumption that $\lim_{\omega \downarrow \omega_{n-1}^c} V_n'(\omega) = V_{n-1}'(\omega_{n-1}^c)$. In general we have

$$\begin{aligned}
\lim_{\omega \downarrow \omega_n^c} V_{n+1}'(\omega) &= \frac{\beta}{\delta} - \rho(\omega_n^c)\left(1 - \frac{\delta}{\beta} \lim_{\omega \downarrow \omega_{n-1}^c} V_n'(\omega)\right) \\
V_n(\omega_n^c) &= \frac{\beta}{\delta} - \rho(\omega_n^c)\left(1 - \frac{\delta}{\beta} V_{n-1}'(\omega_{n-1}^c)\right).
\end{aligned}$$

By the recursive assumption, these two equations are equal and hence, we conclude that $V(\omega)$ is differentiable. In particular, repeated substitution gives

$$V_n'(\omega_n^c) = \frac{\beta}{\delta} - \left(\frac{\delta}{\beta}\right)^{n-1} \prod_{j=1}^{n} \rho(\omega_j^c).$$

Since $\rho(\omega_j^c) \in [(\beta/\delta), \lambda^{-1})$, taking the limit as $n \to \infty$, equivalently $\omega \to \bar{\omega}$, gives $\lim_{\omega \to \bar{\omega}} V'(\omega) = -\infty$. $\square$

Appendix B. Derivation of Risk Measures

Entropy Measure and Proof of Proposition 5

For each $x_t = (\omega_t, s_t)$ and $x_{t+1} = (f(\omega_t, s_t), s_{t+1})$, the price of a k period bond is $p^k(x_t) = \mathbb{E}_t [m(x_t, x_{t+k})]$, which can be defined recursively as:

$$p^k(x_t) = \sum_{s_{t+1}} q(x_t, x_{t+1}) p^{k-1}(x_{t+1}),$$

where $p^0(x_t) = 1$ and state prices are $q(x_t, x_{t+1}) = \pi(s_{t+1}) m(x_t, x_{t+1})$. Let the risk-adjusted probability be defined as $\varrho(x_t, x_{t+k}) = q(x_t, x_{t+k})/p^k(x_t)$. The conditional entropy over the period t to $t+k$ is:

$$L^k(x_t) = -\mathbb{E}_t \left[\log \left(\varrho(x_t, x_{t+k})/\pi(s_{t+k})\right)\right]. \tag{27}$$

Risk-adjusted probabilities and pricing kernel are connected by the following relation: $m(x_t, x_{t+k}) = \pi(s_{t+k}) q(x_t, x_{t+k}) = p^k(x_t) \varrho(x_t, x_{t+k})/\pi(s_{t+k})$. Hence, equation (27) is equivalent to:

$$L^k(x_t) = \log \left(\mathbb{E}_t \left[m(x_t, x_{t+k})\right]\right) - \mathbb{E}_t \left[\log \left(m(x_t, x_{t+k})\right)\right]. \tag{28}$$

Given that $m(x_t, x_{t+k}) = \prod_{j=1}^{k} m(x_{t+j-1}, x_{t+j})$, then equation (28) can also be written as:

$$L^k(x_t) = \log \left(\mathbb{E}_t \left[m(x_t, x_{t+k})\right]\right) - \mathbb{E}_t \left[\sum_{j=1}^{k} \log \left(m(x_{t+j-1}, x_{t+j})\right)\right]. \tag{29}$$

Taking the expectation based on the stationary distribution yields the mean entropy, which we normalize by the time horizon:

$$I(k) := k^{-1} \mathbb{E} \left[L^k(x_t)\right]. \tag{30}$$

Using (29) and define $y^k(x_t) := -(1/k) \log(p^k(x_t))$ the continuously compounded yield of a k period bond, then the mean entropy can be written as:

$$I(k) = -\mathbb{E} \left[y^k(x_t)\right] - \mathbb{E} \log \left(m(x_t, x_{t+1})\right).$$

Risk Measures under Geometric Distribution

Here we derive an explicit formulation of risk measures using the formulas of pricing kernels (21) and (22) provided in the proof of Proposition 6 for $x = (\omega_0^{(n)}, s)$ and $x' = (f(\omega_0^{(n)}, s), s')$ with $n = 0, 1, 2...$. In particular, we derive the (one-period) conditional entropy and the insurance coefficient as functions of the Lagrangian multiplier $\upsilon^{(n)} := 1 + \mu^{(n)}$. We show that these two risk measures are cardinally equivalent.

Conditional Entropy. The conditional entropy over one period is defined as:

$$L^1(x) = \log\left(\mathbb{E}\left[m(x,x')|x\right]\right) - \mathbb{E}\left[\log\left(m(x,x')\right)|x\right].$$

In the geometric distribution case, we have:

$$\begin{aligned} L^1(\omega_0^{(n)}, 2) = \log &\left(\pi \frac{e(2)}{e(1)}\left(\tfrac{\beta}{\delta} + \upsilon^{(0)}\right) + (1-\pi)\left(\tfrac{\beta}{\delta}+1\right)\right) \\ &- \left(\pi \log\left(\frac{e(2)}{e(1)}\left(\tfrac{\beta}{\delta} + \upsilon^{(0)}\right)\right) + (1-\pi)\log\left(\tfrac{\beta}{\delta}+1\right)\right) \end{aligned} \tag{31}$$

and

$$\begin{aligned} L^1(\omega_0^{(n)}, 1) = \log &\left(\pi\left(\frac{\beta}{\delta}\upsilon^{(n)} + \upsilon^{(n+1)}\right) + (1-\pi)\frac{e(1)}{e(2)}\left(\tfrac{\beta}{\delta}\upsilon^{(n)} + 1\right)\right) \\ &- \left(\pi \log\left(\frac{\beta}{\delta}\upsilon^{(n)} + \upsilon^{(n+1)}\right) + (1-\pi)\log\left(\frac{e(1)}{e(2)}\left(\tfrac{\beta}{\delta}\upsilon^{(n)}+1\right)\right)\right). \end{aligned} \tag{32}$$

Inspecting equations (31) and (31), the equilibrium property of $\upsilon^{(n)}$ and Jensen's inequality imply that $L^1(\omega_0^{(n)}, s)$ is larger than zero for each s, invariant in n when $s = 2$, and an increasing function of n when $s = 1$, as shown in Panel (b) of Figure 8.

Insurance Coefficient. The insurance coefficient is defined as:

$$\iota(x) = 1 - \frac{cov[\Delta\log(c^y(x)/e(s)), \Delta\log(e^y(s)/e(s))]}{var[\Delta\log(e^y(s)/e(s))]}.$$

In the geometric distribution case, we have

$$\iota\left(\omega_0^{(n)}, 1\right) = 1 - \frac{\log\left(\frac{c^y(\omega_0^{(n+1)},1)}{e(1)}\right) - \log\left(\frac{c^y(\omega_0^{(n+1)},2)}{e(2)}\right)}{\log\left(\frac{e^y(1)}{e(1)}\right) - \log\left(\frac{e^y(2)}{e(2)}\right)} = 1 - \frac{\log\left(\frac{\upsilon^{(n+1)} + \frac{\beta}{\delta}\upsilon^{(n)}\upsilon^{(n+1)}}{\upsilon^{(n+1)} + \frac{\beta}{\delta}\upsilon^{(n)}}\right)}{\log\left(\frac{e^y(1)}{e(1)}\right) - \log\left(\frac{e^y(2)}{e(2)}\right)} \tag{33}$$

and

$$\iota\left(\omega_0^{(n)},2\right)=1-\frac{\log\left(\frac{c^y(\omega_0^{(0)},1)}{e(1)}\right)-\log\left(\frac{c^y(\omega_0^{(0)},2)}{e(2)}\right)}{\log\left(\frac{e^y(1)}{e(1)}\right)-\log\left(\frac{e^y(2)}{e(2)}\right)}=1-\frac{\log\left(\frac{\upsilon^{(0)}\left(1+\frac{\beta}{\delta}\right)}{\upsilon^{(0)}+\frac{\beta}{\delta}}\right)}{\log\left(\frac{e^y(1)}{e(1)}\right)-\log\left(\frac{e^y(2)}{e(2)}\right)} \tag{34}$$

Inspecting equations (33) and (34), the equilibrium property of $\upsilon^{(n)}$ implies that $\iota(\omega_0^{(n)},s)$ is invariant in n for each $s=2$ and decreasing in n when $s=1$. Since a lower level of insurance coefficient implies less risk sharing, changes in the insurance coefficient are consistent with changes in the conditional entropy or the price of risk.

Appendix C: Shooting Algorithm

In Example 2 of Section 7, the multiplier on the participation constraint in state 2 satisfies $\mu(\omega, 2) = 0$. Therefore, write $\mu(\omega) := \mu(\omega, 1)$. At the stationary solution, write $\mu^{(n)} = \mu(\omega^{(n)})$. We will write $\upsilon^{(n)} := 1 + \mu^{(n)}$. Using the updating condition $\nu(\omega^{(n+1)}) = \mu(n)$, equation (15) can be rewritten as:

$$\begin{aligned}\frac{e(1) - c^y(\omega^{(n)}, 1)}{c^y(\omega^{(n)}, 1)} &= \frac{\beta}{\delta}\left(\frac{\upsilon^{(n-1)}}{\upsilon^{(n)}}\right),\\ \frac{e(2) - c^y(\omega^{(n)}, 2)}{c^y(\omega^{(n)}, 2)} &= \frac{\beta}{\delta}\upsilon^{(n-1)}.\end{aligned}$$

Given Assumption 4, the participation constraint of the young in state 1 and the promise-keeping constraint are binding. That is,

$$\begin{aligned}\pi \log\left(\tfrac{\beta\upsilon^{(n-1)}e(1)}{\beta\upsilon^{(n-1)}+\delta\upsilon^{(n)}}\right) + (1-\pi)\log\left(\tfrac{\beta\upsilon^{(n-1)}e(2)}{\beta\upsilon^{(n-1)}+\delta}\right) &= \omega^{(n)},\\ \log\left(\tfrac{\delta\upsilon^{(n)}e(1)}{\beta\upsilon^{(n-1)}+\delta\upsilon^{(n)}}\right) + \beta\omega^{(n+1)} &= \log(e^y(1)) + \beta\omega_{\min},\end{aligned}$$

for $n \geq 0$ where $\upsilon^{(-1)} = 1$. For $n = 0$

$$\pi \log\left(\tfrac{\beta e(1)}{\beta+\delta\upsilon^{(0)}}\right) + (1-\pi)\log\left(\tfrac{\beta e(2)}{\beta+\delta}\right) = \omega_0,$$

while for n that tends to infinity

$$\pi \log\left(\tfrac{\beta e(1)}{\beta+\delta}\right) + (1-\pi)\log\left(\tfrac{\beta\upsilon_{(\infty)}e(2)}{\beta\upsilon_{(\infty)}+\delta}\right) = \bar{\omega}, \tag{35}$$

where $\upsilon_{(\infty)} = \lim_{n\to\infty}\upsilon_{(n)}$ and $\bar{\omega}$

$$\bar{\omega} = \frac{1}{\beta}\left(\log(e^y\,(1)) - \log\left(\tfrac{\beta e(1)}{\beta+\delta}\right)\right) + \pi\log(e^o\,(1)) + (1-\pi)\log(e^o\,(2)). \tag{36}$$

Replacing equation (36) into (35), we have:

$$\upsilon^{(\infty)} = \frac{\delta}{\beta}\left(-1 + \left(\left(\tfrac{\beta+\delta}{\delta}\right)^{\frac{1+\beta\pi}{\beta(1-\pi)}}\left(\tfrac{e^y(1)}{e(1)}\right)^{\frac{1}{\beta(1-\pi)}}\left(\tfrac{e^o(1)}{e(1)}\right)^{\frac{\pi}{1-\pi}}\tfrac{e^o(2)}{e(2)}\right)^{-1}\right)^{-1}. \tag{37}$$

which is independent of θ. Using the above equations, we can derive a second-order difference equation for $\upsilon^{(n)}$ where

$$\upsilon^{(n+1)} = \frac{\beta}{\delta}\upsilon^{(n)} \left(-1 + \left(\frac{\beta\upsilon^{(n)}}{\beta\upsilon_{(n)}+\delta}\right)^{\frac{1-\pi}{\pi}} \left(\frac{\beta\upsilon^{(\infty)}+\delta}{\beta\upsilon^{(\infty)}}\right)^{\frac{1-\pi}{\pi}} \left(\frac{\beta+\delta}{\delta}\right)^{\frac{1}{\beta\pi}} \left(\frac{\beta+\delta}{\beta}\right) \left(1 + \frac{\beta}{\delta}\frac{\upsilon^{(n-1)}}{\upsilon^{(n)}}\right)^{-\frac{1}{\beta\pi}} \right) \tag{38}$$

It can be shown that the second-order difference equation in (38) has a unique saddle path solution. The solution can be found by a *forward shooting* algorithm to find the value of $\upsilon^{(0)}$ such that in the difference equation (38), the absolute difference between $\upsilon^{(N)}$ and υ^{∞} given in equation (37) is sufficiently close for N sufficiently large.

Appendix D: Pseudo-code for Numerical Algorithm

Algorithm Find Value and Policy Functions

```
procedure                                   ▷ Find solution to functional equation (14)
    Ω ← [ω_min, ω̄]                           ▷ ω_min and ω̄ computed
    gridpoints ← gp                          ▷ Discretize Ω: gp = 150 Chebyshev interpolation points
    tolerance ← ϵ > 0                        ▷ ϵ = 10^-6
    J ← V*                                   ▷ V* is First-best
    repeat
        Compute TJ from J                    ▷ Use equation (14) and interpolate
        d ← d(TJ, J)                         ▷ d(TJ, J) = max_ω|TJ(ω) − J(ω)|
        J ← TJ
    until d < ϵ
    V ← J
    Compute f(ω, s) and g(ω, s)              ▷ Using the function V just computed.
end procedure
```

Algorithms are implemented in MATLAB®. At each iteration, the optimization uses the nonlinear programming solver command `fmincon`. Interpolation of the value function uses the spline method embedded in the `interp1` command. In a typical example, the value function converges within 300 iterations. Each iteration takes about 1 second on a standard pc.